Minimum-Phase Robustness for Second-Order Linear Systems

Jong-Lick Lin and Shin-Ju Chen
National Cheng-Kung University, Tainan 701, Taiwan, Republic of China
and Jer-Nan Juang
NASA Langley Research Center, Hampton, Virginia 23681-0001

A novel approach is presented to address the problem of minimum-phase robustness for second-order linear systems with uncertainties in the input influence matrix. Two cases are studied, including unstructured uncertainties and unidirectional perturbation of the input matrix. A tolerable margin is given for unstructured uncertainties in terms of the structured singular value to guarantee minimum phase for the systems. For a specified unidirectional perturbation, the exact bounds can be determined by examining the eigenvalue loci of a certain rational function. The approach is applicable to the systems with noncollocated actuators and sensors. To illustrate the concept several numerical examples are provided.

Introduction
The minimum-phase property is an important issue for dealing with synthesis problems in control systems, such as typical adaptive control,\(^1\) passive control,\(^2,3\) and optimal feedback control.\(^4\) In some areas of robust control theory, the minimum-phase property is required for uncertain systems such that the robustness of stability can be guaranteed within the variation ranges of uncertainties.\(^5,6\)

Thus, the minimum-phase property is very important for the control systems. The minimum-phase system is defined as the system having none of its transmission zeros on the right-hand side of the complex plane. Some application of transmission zeros in control theory are given in Refs. 7–10. It has been proven\(^10\) that the transmission zeros of a structure with collocated sensors and actuators are closely related to poles, i.e., the natural frequencies and damping ratios of the structure. The sensitivities of the transmission zeros relative to perturbations of system parameters have also been proven\(^11\) to be closely related to the sensitivities of the poles. The effect of model order uncertainty to the transmission zeros was known\(^12,13\) to have some influence on the pole/zero cancellation controller designed for a truncated model of a given large flexible structure. Studies of the transmission zeros of flexible structures were also extended in Ref. 14 to the case of noncollocated sensors and actuators. The conditions for a system, represented by a finite-dimensional model, to be minimum phase were derived in Ref. 15 for the case of noncollocated sensors and actuators.

For a given uncertain linear system model described by first-order dynamic equations, the minimum-phase robustness problem is transformed to the stability robustness problem of generalized eigenvalues with respect to unstructured or structured uncertainties.\(^16,17\) However, for a second-order system, transforming to a first-order form not only increases the dimension of the problem, but also destroys the sparsity of the structural matrices. Not only physical insight but computational efficiency is often lost in conversion to a first-order system. As a result, the minimum-phase problem will be addressed in this paper directly using the representation of a second-order equation.

It is well known that a second-order linear system with collocated sensors and actuators is minimum phase.\(^10\) Mathematically, the collocated sensors and actuators here mean that the input influence matrix is equal to the transpose of the output influence matrix. It should be noted that physically collocated sensors and actuators do not necessarily have the mathematical collocation as stated earlier. The minimum-phase property can be maintained for the noncollocated case if certain conditions are satisfied. As shown in Ref. 15, if the columns of the input matrix are in the column space generated by the transpose of the output matrix, then the second-ordersystem is ensured to be minimum phase. One question that one may ask is how robust the minimum phase is relative to the input matrix uncertainties.

The objective is to study the minimum-phase robustness problem subject to input matrix uncertainties for second-order linear systems that are nominally minimum phase with properly located sensors and actuators. Some conditions are presented to determine the maximal uncertainty bounds of input influence matrix that the system may tolerate to maintain the minimum-phase property. Two kinds of input matrix uncertainties are considered. One is the unstructured uncertainty and the other is the unidirectional perturbation that belongs to a subset of structured uncertainties. The structured singular-value-analysis technique\(^18–22\) is used to find the bounds for the unstructured uncertainty, whereas the eigenvalue loci method is used to determine the maximum bounds for the unidirectional perturbation. Illustrative examples are given to show the feasibility of the proposed technique.

Conventional Approach
Consider the \(n\)-order, \(m\)-output, \(r\)-input, linear time-invariant system described by

\[
Mw + Zw + Kw = Bu
\]

\(y = Cw\)

with \(M(n \times n) \geq 0\), \(Z(n \times n) \geq 0\), and \(K(n \times n) \geq 0\), where \(w(n \times 1)\) is the state vector, \(u(r \times 1)\) the input vector, \(B(n \times r)\) the input influence matrix, \(y(m \times 1)\) the output vector, and \(C(m \times n)\) the output influence matrix. The Laplace transform of Eq. (1) is

\[
\begin{bmatrix}
N(s) \\
C
\end{bmatrix} \begin{bmatrix}
B \\
0
\end{bmatrix} \begin{bmatrix}
w(s) \\
-u(s)
\end{bmatrix} = \begin{bmatrix}
0 \\
y(s)
\end{bmatrix}
\]

where \(s\) is the variable of the Laplace transform, \(0\) inside a matrix is a zero matrix or a zero vector with an appropriate dimension, and

\[
N(s) := Ms^2 + Zs + K
\]

Assume that the system is completely controllable and completely observable. The transmission zeros of system (1) are defined as the
set of complex number $s$ such that the following equations
\[
\begin{bmatrix}
N(s) & B \\
C & 0
\end{bmatrix}
\begin{bmatrix}
\tilde{w}(s) \\
\tilde{u}(s)
\end{bmatrix}
= \begin{bmatrix}
0 \\
0
\end{bmatrix}
\] (4)

have a nonzero solution vector
\[
\begin{bmatrix}
\tilde{w} \\
\tilde{u}
\end{bmatrix}
\]
In other words, the leftmost matrix of Eq. (2) loses its column rank, i.e.,
\[
\text{rank}\left\{\begin{bmatrix}
N(s) & B \\
C & 0
\end{bmatrix}\right\} < n + r
\] (5)

Without loss of generality, assume that the input matrix $B$ has full column rank $r$, where $r \leq n$, and the output matrix $C$ has full row rank $m$. If Eq. (4) does not have a nonzero vector
\[
\begin{bmatrix}
\tilde{w} \\
\tilde{u}
\end{bmatrix}
\]
for the complex variable $s$ with $\text{real}[s] > 0$, the corresponding system is said to be minimum phase, i.e., all of the transmission zeros are in the left-hand complex plane (LHCP).

Equation (4) is a second-order differential equation with constraint $\tilde{C}\tilde{w} = 0$. Mathematically, it is a constrained second-order differential equation that may be rearranged equivalently into an unconstrained second-order differential equation as follows. Because the $m \times n$ matrix $C$ has full row rank, there exists a constant $n \times n(m \leq n)$ permutation matrix $P = [P_1, P_2]$ such that $CP_1$ is invertible and
\[
CP = [CP_1, CP_2] = [\tilde{C}_1, \tilde{C}_2]
\] (6)

where $\tilde{C}_1 := CP_1$ and $\tilde{C}_2 := CP_2$. The $m \times m$ matrix $\tilde{C}_1$ is nonsingular with rank $m$. Note that a permutation matrix $P$ must be nonsingular with $P^{-1} = PT$, and hence, a transformation can be defined as
\[
z := P^{-1}\tilde{w}
\] (7)

Let the $n \times 1$ vector $z$ be partitioned in the form
\[
z = \begin{bmatrix}
z_1 \\
z_2
\end{bmatrix}
\] (8)

Then, by Eqs. (6–8), we have
\[
\begin{align*}
\tilde{C}\tilde{w} &= CPz \\
&= [\tilde{C}_1, \tilde{C}_2] \begin{bmatrix}
z_1 \\
z_2
\end{bmatrix} \\
&= \tilde{C}_1z_1 + \tilde{C}_2z_2
\end{align*}
\] (9)

that in turn yields
\[
\tilde{C}\tilde{w} = 0 \iff \tilde{C}_1z_1 + \tilde{C}_2z_2 = 0
\] (10)
or, equivalently,
\[
z = \begin{bmatrix}
z_1 \\
z_2
\end{bmatrix} = -\tilde{C}_1^{-1}\tilde{C}_2z_2
\] (11)

where $I_{n \times n}$ is an $(n-m) \times (n-m)$ identity matrix. Moreover, in view of Eqs. (7), (8), and (10), one obtains
\[
\begin{align*}
\tilde{w} &= Pz = P_1z_1 + P_2z_2 \\
&= (P_2 - P_1\tilde{C}_1^{-1}\tilde{C}_2)z_2 \\
&=: \Pi z_2
\end{align*}
\] (12)

where the $n \times (n-m)$ matrix $\Pi$ is defined as $\Pi := (P_2 - P_1\tilde{C}_1^{-1}\tilde{C}_2)$, and thus, Eq. (1a) can be rewritten as
\[
\dot{\tilde{u}} = (\tilde{M}s^2 + \tilde{Z}s + \tilde{K})\tilde{w}
\] (13)

For simplicity, let us define
\[
\tilde{M} := MP_1, \quad \tilde{Z} := ZP_1, \quad \tilde{K} := KP_1
\] (14)

where $\tilde{M}$, $\tilde{Z}$, and $\tilde{K}$ are all $n \times (n-m)$ matrices. With these notations, Eq. (13) becomes
\[
(\tilde{M}s^2 + \tilde{Z}s + \tilde{K})z_2 = B\tilde{u}
\] (15)

This is an unconstrained second-order differential equation that is mathematically equivalent to the constrained differential equation shown in Eq. (4). Equation (15) can be rearranged to yield a matrix equation of the form
\[
[\tilde{M}s^2 + \tilde{Z}s + \tilde{K}]z_2 = B\tilde{u}
\] (16)

If Eq. (16) does not have any nonzero solution for all $s$ in right-hand complex plane (RHCP), then the system described by Eq. (4) is minimum phase. In other words, if the columns of the $n \times (n-m+r)$ matrix $[\tilde{M}s^2 + \tilde{Z}s + \tilde{K} : B]$ are linearly independent for all $s \in \text{RHCP}$, then the zero vector is the only solution of Eq. (16), and thus the system is minimum phase. For the case where the number of outputs is greater than or equal to that of inputs, namely, $m \geq r$, the matrix $[\tilde{M}s^2 + \tilde{Z}s + \tilde{K} : B]$ has fewer columns than rows. Therefore, it becomes clear that the system, Eq. (4), is minimum phase if and only if
\[
[\tilde{M}s^2 + \tilde{Z}s + \tilde{K}] \text{ has full column rank, for all } s \in \text{RHCP}
\] (17)

for the case where $m \geq r$.

For the following derivation, let us define the $n \times (n-m)$ matrix $\tilde{N}(s)$ as
\[
\tilde{N}(s) := \tilde{M}s^2 + \tilde{Z}s + \tilde{K} = N(s)\Pi
\] (18)

We have studied the minimum-phase robustness based on the fact that if both matrices $B$ and $C$ have full rank, and $B = C^T\Gamma$ for any chosen $m \times r$ real matrix $\Gamma$, then the system described by Eq. (1) or equivalently Eq. (4) is minimum phase. In the following section, we will address the robustness of the minimum-phase property relative to uncertainty in the input influence matrix.

**Input Influence Matrix with Unstructured Perturbation**

Here is the question: To what extent of the perturbation $\Delta_B$ away from the nominal input matrix $B_0 = C^T\Gamma$ is the system (4) still guaranteed to be minimum phase? Mathematically, the perturbed input matrix $B$ can be represented by
\[
B = B_0 + \Delta_B
\] (19)

where the $n \times r$ matrix $\Delta_B$ is an additive real uncertainty of matrix $B$. Define the $n \times (n-m+r)$ matrix $H(s)$ as
\[
H(s) := [\tilde{M}s^2 + \tilde{Z}s + \tilde{K} : B]
\]

where matrices $H_0(s) := [\tilde{N}(s) : B_0] \in C^{+ \times (n-m+r)}$ and $\Delta_H := [0 : \Delta_B] \in R^{+ \times (n-m+r)}$ denote the nominal and perturbed parts of the matrix $H(s)$, respectively.

Remark: It has been proven$^3$ from Eq. (16) that if $B = B_0$ then system (1) is minimum phase, i.e., Eq. (16) does not have any nonzerosolutionfor all $s \in \text{RHCP}$. Therefore $H_0(s) = [\tilde{M}s^2 + \tilde{Z}s + \tilde{K} : B_0]$ is an $n \times (n-m+r)$ matrix of full column rank for all $s \in \text{RHCP}$.

The matrix rank is unchanged upon left multiplication by a matrix of its complex conjugate transpose,$^3$ namely,
\[
\text{if } A \in C^{m \times n}, \quad \text{then } \text{rank}(A^*A) = \text{rank}(A)
\] (21)

where $C^{m \times n}$ denotes the set of all $m$-by-$n$ complex matrices.
From Eq. (21) in conjunction with the equality, $\Delta_H^* = \Delta_H^T$, one obtains

$$\text{rank}[H(s)] = \text{rank}[H_0(s) + \Delta_H]$$

$$= \text{rank}\left[H_0(s) + (H_0(s) + \Delta_H)^* \cdot (H_0(s) + \Delta_H)\right]$$

$$= \text{rank}[H_0^*(s)H_0(s) + H_0^T(s)\Delta_H + \Delta_H^TH_0(s) + \Delta_H^T\Delta_H]$$

(22)

for any fixed $s \in C$. Because $H_0(s)$ is an $n \times (n - m + r)$ matrix of full column rank for all $s \in \mathbb{R}$, and thus, the square matrix, $H_0^*(s)H_0(s)$ is nonsingular for all $s \in \mathbb{R}$, as a result, Eq. (22) can be rewritten as

$$\text{rank}[H_0^*(s)H_0(s) + H_0^T(s)\Delta_H + \Delta_H^TH_0(s) + \Delta_H^T\Delta_H]$$

(23)

where $Q(s) := [H_0^*(s)H_0(s)]^{-1}$ is a nonsingular matrix with dimension $(n - m + r)$. Hence, from Eqs. (17), (20), and (23), the system, Eq. (1), is minimum phase if and only if $H(s)$ has full column rank for all $s \in \mathbb{R}$.

$$I_{n-m+r} + Q(s)H_0^*(s)\Delta_H + Q(s)\Delta_H^TH_0(s) + Q(s)\Delta_H^T\Delta_H$$

is nonsingular, $\forall s \in \mathbb{R}$, or

$$\text{det}[I_{n-m+r} + Q(s)H_0^*(s)\Delta_H + Q(s)\Delta_H^TH_0(s) + Q(s)\Delta_H^T\Delta_H] \neq 0, \quad \forall s \in \mathbb{R}$$

On the other hand, the system becomes nonminimum phase, if there exists an uncertainty matrix $\Delta_A$ such that

$$\text{det}[I_{n-m+r} + Q(jw)H_0^*(jw)\Delta_A + Q(jw)\Delta_A^TH_0(jw)]$$

(24)

for some $w \in \mathbb{R}$. Consequently, the minimum-phase robustness can be described as follows: Find the maximum size of uncertainty $\Delta_A$ such that the system represented by Eq. (1) still maintains minimum phase.

Using the equality

$$\text{det}(I + UV) = \text{det}(I + VU)$$

(25)

for any matrices $U$ and $V$ of appropriate dimensions yields

$$\text{det}[I_{n-m+r} + Q(jw)H_0^*(jw)\Delta_A + Q(jw)\Delta_A^TH_0(jw)]$$

$$+ \text{det}[I_{n-m+r} + Q(jw)H_0^*(jw)\Delta_A + Q(jw)\Delta_A^TH_0(jw)]$$

$$= \text{det}[I_{n-m+r} + Q(jw)H_0^*(jw)\Delta_A + Q(jw)\Delta_A^TH_0(jw)]$$

(26)

Recall that $\Delta_H = [0: \Delta_A]$ and thus, Eq. (26) can be readily reduced to a simpler form in terms of matrix $\Delta_A$. For a $(2n - m + r) \times (2n - m + r)$ matrix

$$\begin{bmatrix}
\Delta_H & 0 \\
0 & \Delta_H^T
\end{bmatrix}$$

(27)

there exists matrices $L$ and $R$ of appropriate dimensions such that

$$\begin{bmatrix}
0 & \Delta_B : 0 \\
\ldots & \ldots \\
0 & 0 \\
0 & \Delta_T
\end{bmatrix} = \begin{bmatrix}
L & R
\end{bmatrix}$$

(28)

where the block diagonal matrix $\Delta_B$ is represented by

$$\Delta_B = \text{diag}(\Delta_{b11}I_2, \Delta_{b12}I_2, \ldots, \Delta_{b1r}I_2, \Delta_{b21}I_2, \Delta_{b22}I_2, \ldots, \Delta_{bnr}I_2)$$

(29)

and

$$L \in \mathbb{R}^{(2n - m + r) \times (2nr)}, \quad R \in \mathbb{R}^{(2nr) \times (2n - m + r)}$$

Here $\mathbb{R}^{l \times j}$ denotes the set of all $l$-by-$j$ real matrices. The significance of this approach is demonstrated in the following example:

**Example 1:** Consider a mechanical system with three states, two inputs, and two outputs, namely, $n = 3, m = 2$, and $r = 2$. It is clear that

$$\begin{bmatrix}
0 & \Delta_B : 0 \\
\ldots & \ldots \\
0 & 0 \\
0 & \Delta_T
\end{bmatrix} = \begin{bmatrix}
0 & \Delta_{b11}I_2 & 0 & 0 & 0 \\
0 & \Delta_{b12}I_2 & 0 & 0 & 0 \\
0 & 0 & \Delta_{b21}I_2 & 0 & 0 \\
0 & 0 & \Delta_{b22}I_2 & \Delta_{b31} \Delta_{b32} \Delta_{b33}
\end{bmatrix}$$

(30)

Mathematical manipulation then yields

$$L = \begin{bmatrix}
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}$$

(32)

$$R = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}^T$$

(33)
where the $2n_r \times 2n_r$ square matrix $T(j\omega)$ is defined as

$$
T(j\omega) := R \begin{bmatrix} L_{n - m + 1, n} & 0 \\ H_0(j\omega) & I_n \end{bmatrix} \begin{bmatrix} Q(j\omega) & 0 \\ 0 & I_n \end{bmatrix} L
$$

Thus, the minimum-phase robustness problem can be mathematically rewritten as

$$
\min_{\omega \in \mathbb{R}} [\Delta(\omega) : \text{det}(L_{n, m} + T(j\omega)\Delta(\omega)) = 0] 
$$

where $\Delta(\omega)$ is a block diagonal real matrix shown in Eq. (29) and $\hat{\sigma}(\cdot)$ is the maximum singular value. In view of this optimization problem, we turn our attention to the notion of structured singular value that is proposed to measure robustness in Eq. (30) to define the structured singular value with respect to the underlying block structure of uncertainties, let $G \in \mathbb{C}^{n \times n}$ and $m_i, m_j, \ldots$ be three nonnegative integers with $m = m_i + m_j + \ldots \leq n$. The block structure $K(m_i, m_j, \ldots)$ is an $m$-tuple of positive integers

$$
K = \left\{ k_1, \ldots, k_m, k_{m+1}, \ldots, k_{m+n}, \ldots, k_{m+n-1}, \ldots, k_m \right\}
$$

where

$$
\sum_{i=1}^{m} k_i = n
$$

Now define the set of allowable uncertainties

$$
X_K = \left\{ \text{diag}\left[\delta_1 I_{k_1}, \ldots, \delta_m I_{k_m}, \delta_{m+1} I_{k_{m+1}}, \ldots, \delta_{m+n} I_{k_{m+n}}, \ldots, \delta_{m+n-1} I_{k_{m+n-1}}, \ldots, \delta_m I_{k_m} \right] \right\}
$$

which involves repeated real scalars, repeated complex scalars, and full complex block. The purely complex case corresponds to $m = 0$ and the purely real case to $m = m_j = 0$.

**Definition 1 (Ref. 19):** For a complex matrix $G \in \mathbb{C}^{m \times n}$, the structured singular value of $G$ with respect to a block structure $K(m_i, m_j, \ldots)$ is the number defined such that $\mu_K(G)$ is equal to the smallest $\hat{\sigma}(\Delta)$ needed to make $(I - G\Delta)$ singular. That is,

$$
\mu_K(G) := \min_{\Delta \in \mathbb{R}^{m \times n}} [\hat{\sigma}(\Delta) : \text{det}(I - G\Delta) = 0]
$$

If no $\Delta \in X_K$ exists such that $\text{det}(I - G\Delta) = 0$, then $\mu_K(G) := 0$.

The computation of the structured singular value is not easy, only upper and lower bounds are available for the real/mixed $\mu$ problem. The software used to calculate the real/mixed $\mu$ is currently available, based on the work in the toolbox of MATLAB. Now, based on the definition of structured singular value in Eq. (38), the optimization problem of minimum-phase robustness in Eq. (35) can be addressed in the following theorem.

**Theorem 1:** Under the assumption that the nominal system, Eq. (1), with $B = B_0 \in \mathbb{C}^{2 \times 1}$ is minimum phase, then the second-order system with the input influence matrix $B = B_0 + \Delta_B$, where $\Delta_B \in \mathbb{R}^{2 \times 1}$ preserves its minimum phase property, if

$$
|\Delta_{kj}| < \left\{ \sup_{\omega \in \mathbb{R}} [\mu_{\Delta}(T(j\omega))] \right\}^{-1}
$$

**Proof:** It is apparent that the smallest size of $\Delta_B$, which makes the system, Eq. (1), become nonminimum phase, is equal to

$$
\left\{ \sup_{\omega \in \mathbb{R}} [\mu_{\Delta}(T(j\omega))] \right\}^{-1}
$$

The range of the elements $\Delta_{kj}$ in $\Delta_B$ is then determined.

To this end, we have been concerned with the unstructured perturbation $\Delta_B$ of input influence matrix $B = B_0 + \Delta_B, \Delta_B \in \mathbb{R}^{2 \times 1}$, under the fact that the nominal system, Eq. (1), with $B = B_0 = C\gamma$ is minimum phase. The perturbation term $\Delta_B$ is assumed to be arbitrary. For a physical system, $\Delta_B$ may not be arbitrarily unstructured because of physical constraints. Some special forms of $\Delta_B$ will be considered and studied in the following section.

**Input Influence Matrix with Unidirectional Perturbation**

For a given mechanical system composed of masses, springs, and dashpots, the input influence matrix $B$ is determined by the locations of actuators, and thus the perturbation $\Delta_B$ cannot be arbitrarily unstructured. It may be a specified matrix of unidirectional perturbation that belongs to a subset of structured uncertainties, and is

$$
B = B_0 + \Delta_B = B_0 + kU
$$

The following example is provided for readers to understand this point.

**Example 2:** Consider a simple mass–spring–dashpot system shown in Fig. 1.

The dynamical equation is described as a second-order system represented by Eq. (1) with

$$
\begin{bmatrix} \dot{w}_1 \\
\dot{w}_2 \\
\end{bmatrix} = \begin{bmatrix} 1 & 0 \\
0 & 1 \\
\end{bmatrix} \begin{bmatrix} m_1 & 0 \\
0 & m_2 \\
\end{bmatrix} \begin{bmatrix} w_1 \\
w_2 \end{bmatrix} + \begin{bmatrix} k_1 & -k_1 \\
-k_1 & k_1 \\
\end{bmatrix} u + \begin{bmatrix} b_1 & -b_1 \\
-b_1 & b_1 \end{bmatrix} \begin{bmatrix} u_1 \\
u_2 \end{bmatrix}
$$

If the action forces $u_1$ and $u_2$ exerted by the actuators are only on masses $m_1$ and $m_2$, then there is no difference in system matrices except the input influence matrix replaced by

$$
B_1 = \begin{bmatrix} 1 & 0 \\
0 & 0 \\
0 & 1 \\
\end{bmatrix}
$$

In comparison with the original input influence matrix $B_0$, the difference between $B_0$ and $B_1$ can be viewed as an unidirectional perturbation $kU$ with $k = 1$, namely,

$$
U_1 = B_1 - B_0 = \begin{bmatrix} 1 & 0 \\
0 & 0 \\
0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\
0 & 0 \\
0 & 1 \\
\end{bmatrix} \begin{bmatrix} 0 & 0 \\
0 & 0 \\
0 & 0 \\
\end{bmatrix}
$$
In addition, if the action forces \( v_1 \) (on \( m_1 \)) and \( v_2 \) (on \( m_2 \)) exerted by the actuators are changed to 1.2\( v_1 \) and 1.5\( v_2 \), then the input influence matrix and the unidirectional perturbation, respectively, become

\[
B_2 = \begin{bmatrix} 1.2 & 0 \\ 0 & 0 \\ 0 & -1.5 \end{bmatrix}, \quad U_2 = \begin{bmatrix} 0.2 & 0 \\ 1 & 0 \\ 0 & -0.5 \end{bmatrix}
\]

**Remark 2:** In Example 2, it is easy to see that there exists a matrix \( \Gamma \) such that

\[
B_0 = \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = C^T \Gamma
\]

Note that the system is minimum phase because it is collocated with a sign change. The systems with the different input influence matrix \( B_1 \) or \( B_2 \) is still minimum phase if the maximal tolerable bound of the specified unidirectional perturbation \( U_i, i = 1, 2 \), is greater than 1 for the perturbed system, although there exists no matrix \( \Gamma \) satisfying \( B_1 = C^T, \Gamma \) or \( B_2 = C^T, \Gamma \).

The problem to be studied in this section is to what extent of the unidirectional perturbation \( U \) for the system described by Eq. (1) may preserve the minimum-phase property. Similar to Eq. (29), the block diagonal matrix \( \Delta_B \) becomes

\[
\Delta_B = k \text{diag}(U_1, I_2, U_1 I_2, \ldots, U_1 I_2, U_1 I_2, \ldots, U_m I_2) =: k \bar{U}
\]

In consequence, the system becomes nonminimum phase if and only if

\[
\det[I_{\alpha r} + k T(j \omega) \bar{U}] = 0
\]

for some \( \omega \). Let \( \lambda_i(j \omega), 1 \leq i \leq 2n_r \), be the \( i \)th eigenvalue of the complex matrix \( T(j \omega) \bar{U} \), namely, \( \det(\lambda_i(j \omega) I_{\alpha r} - T(j \omega) \bar{U}) = 0 \) for \( \omega \geq 0 \). Because the eigenvalues are continuous functions of the entries of a matrix, \( \lambda_i(j \omega) \) is continuous in \( \omega \) for \( 1 \leq i \leq 2n_r \). If the eigenvalue loci of \( T(j \omega) \bar{U} \) intersect the real axis at \( (\delta, 0) \), there may exist a real number \( k = 1/\delta \) satisfying \( \det[I_{\alpha r} - k T(j \omega) \bar{U}] = 0 \). Let \( k_{\text{min}}^{\text{real}} \) and \( k_{\text{max}}^{\text{real}} \) be the largest negative and the smallest positive real numbers, respectively such that Eq. (41) holds. Mathematically, this can be written as

\[
k_{\text{min}}^{\text{real}} = \max\{k < 0 : \det[(1/k) I_{\alpha r} - T(j \omega) \bar{U}] = 0 \text{ for } \omega \geq 0\}
\]

(42)

and

\[
k_{\text{max}}^{\text{real}} = \min\{k > 0 : \det[(1/k) I_{\alpha r} - T(j \omega) \bar{U}] = 0 \text{ for } \omega \geq 0\}
\]

(43)

They are the inverse of the smallest negative and the largest positive real numbers where eigenvalue loci of \( T(j \omega) \bar{U} \) intersect with real axis, respectively. As a result, the tolerable margin for minimum phase robustness of the system represented by Eq. (1) is determined by the following theorem.

**Theorem 2:** Under the assumption that the nominal second-order system, Eq. (1), with \( B = B_0 = C^T \Gamma \) is minimum phase, then the same second-order system with a perturbed input influence matrix \( B = B_0 + kU \), where \( U \in \mathbb{R}^{n_x \times 1} \) preserves its minimum-phase property if \( k \in (k_{\text{min}}^{\text{real}}, k_{\text{max}}^{\text{real}}) \).

**Remark 3:** If the eigenvalue loci of \( T(j \omega) U \) for \( \omega \geq 0 \) intersect the real axis at points \( (\lambda_1, 0), (\lambda_2, 0), \ldots, (\lambda_{n_r}, 0), (\lambda_{n_r}^*, 0), \) \ldots, and \( (\lambda_{n_r}^*, 0) \) with \( \lambda_1 < \cdots < \lambda_{n_r} < \cdots < \lambda_{n_r}^* < \cdots < \lambda_{n_r}^* \) in the complex plane, then \( k_{\text{min}}^{\text{real}} = 1/\lambda_{n_r} \) and \( k_{\text{max}}^{\text{real}} = 1/\lambda_{n_r}^* \). For the case where the eigenvalue loci do not intersect the negative real axis for all \( \omega \geq 0 \), we define \( k_{\text{min}}^{\text{real}} = -\infty \). On the other hand, if the loci do not intersect the positive real axis, we define \( k_{\text{max}}^{\text{real}} = +\infty \).

**Example 3:** This numerical example is given to illustrate the application of the proposed technique. Let us consider the mechanical system described by Example 2 where

\[
m_1 = 0.71, \quad c_1 = 0.68, \quad k_1 = 0.13, \quad m_2 = 0.45
\]

\[
c_2 = 0.81, \quad k_2 = 0.40, \quad m_3 = 1.07
\]

and, thus,

\[
M = \begin{bmatrix} 0.71 & 0 & 0 \\ 0 & 0.45 & 0 \\ 0 & 0 & 1.07 \end{bmatrix}, \quad Z = \begin{bmatrix} 0.68 & -0.68 & 0 \\ -0.68 & 1.49 & -0.81 \\ 0 & -0.81 & 0.81 \end{bmatrix}
\]

\[
K = \begin{bmatrix} 0.13 & -0.13 & 0 \\ -0.13 & 0.53 & -0.40 \\ -0.40 & 0.40 & 0 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}
\]

\[
\Delta_B = \begin{bmatrix} \Delta_{b11} \\ \Delta_{b12} \\ \Delta_{b21} \\ \Delta_{b22} \end{bmatrix}, \quad C = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]

Because \( B_0 = C^T \Gamma \)

\[
\Gamma = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}
\]

the nominal system is minimum phase. Furthermore, matrix \( C \) is of full row rank, and thus there exists a permutation matrix

\[
P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}
\]

such that

\[
CP = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ \cdots & \cdots & \cdots \end{bmatrix} = [\hat{C}_1 : \hat{C}_2]
\]

where matrix \( \hat{C}_1 \) is nonsingular. From Eq. (12), the matrix \( \Pi \) is given by

\[
\Pi = P_2 - P_1 \hat{C}_1^{-1} \hat{C}_2 = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^T
\]

and from Eq. (14), we have

\[
\hat{M} = M \Pi = \begin{bmatrix} 0.71 \\ 0.45 \\ 0 \end{bmatrix}, \quad \hat{Z} = Z \Pi = \begin{bmatrix} 0 \\ 0.81 \\ -0.81 \end{bmatrix}
\]

\[
\hat{K} = K \Pi = \begin{bmatrix} 0 \\ 0.40 \\ -0.40 \end{bmatrix}
\]

Thus, Eq. (20) yields

\[
H_0(j \omega) = [\hat{N}(j \omega), \hat{B}_0] =
\]

\[
= \begin{bmatrix} -0.71 \omega^2 \\ -0.45 \omega^2 + 0.40 + 0.81 j \omega \\ -0.40 - 0.81 j \omega \end{bmatrix}
\]

The matrices \( L \) and \( R \) in Example 1 are used to diagonalize \( \Delta_B \) in Eq. (29). Substituting matrices \( H_0(j \omega), Q(j \omega) = (H_0(j \omega)^T H_0(j \omega))^{-1} L \) and \( R \) into Eq. (34) yields the matrix \( T(j \omega) \).

The upper bound of

\[
\sup_{\omega \in \mathbb{R}} \mu_{\Delta_B}(T(j \omega)) = 4.9557
\]
by using the $\mu$-toolbox. By Theorem 1, the system preserves the minimum-phase property if

$$|\Delta_{\omega i}| < \frac{1}{4.9557} = 0.2018 \quad \text{for} \quad i = 1, 2, 3; \quad j = 1, 2$$

Example 4: Consider Example 3 again with the perturbation $\Delta_{B} = kU_{1}$ shown in Example 2. The diagonal matrix $\hat{U}$ has the form $\hat{U} = \text{diag}(0, 0, 1, 0, 0, 0)$. The eigenvalue loci of $T(j\omega)\hat{U}$ are shown in Fig. 2. The largest positive real number where the eigenvalue loci of $T(j\omega)\hat{U}$ intersect with the real axis is 0.61206, but the loci do not intersect with the negative real axis. Hence, the maximal minimum-phase interval is

$$[k_{-}^{min}, k_{+}^{max}] = \left[ -\infty, \frac{1}{0.61206} \right] = (-\infty, 1.6338)$$

It is seen from $k_{+}^{max} > 1$ that the system is still minimum phase if $B = B_{1}$, as shown in Example 2, although there exists no matrix $\Gamma$ satisfying $B_{1} = C^{T}\Gamma$. In addition, the margin of unidirectional perturbation, which the minimum-phase system can tolerate, is clearly larger than that corresponding to an unstructured uncertainty in the input influence matrix.

**Conclusions**

The problem of minimum-phase robustness for second-ordersystems has been studied. The uncertainty tolerance for the input influence matrix is determined to guarantee the system to remain minimum phase. The approach used exploits an equivalent relationship between minimum-phase robustness and robust nonsingularity. If the perturbation is unstructured, then real structured singular value analysis is involved in obtaining the uncertainty bounds. However, the exact structured singular value is difficult to find, and thus an upper bound is used; this makes the results conservative. In particular, if the perturbation in the input influence matrix is unidirectional, then the exact bound can be obtained by plotting the eigenvalue loci of a certain rational function matrix. This new approach provides a useful tool for examining the minimum-phase property for any mechanical system that can be described by a second-order matrix differential equation.

**References**