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This paper presents a robust controller design for second-order dynamic systems. The controller is model independent and is a virtual second-order dynamic system. The conditions for actuator and sensor placements are identified for controller designs that guarantee overall closed-loop stability. The dynamic controller can be viewed as a virtual passive damping system that serves to stabilize the actual dynamic system. The control gains are interpreted as virtual mass, spring, and dashpot elements that play the same roles as actual physical elements in stability analysis. Position, velocity, and acceleration feedback are considered. Simple examples are provided to illustrate the controller design. From this illustration the physical meaning of the controller design is apparent.

Introduction

CONTROl theory for time-invariant linear systems that are described by first-order dynamic equations has been well established for decades. Today control software tools are also written in first-order forms. For applications, engineers can simply convert whatever models they have to the first-order forms and then use the existing tools to design the controllers. If the performance requirements are satisfied by the controllers, the design jobs are completed. If not, the design parameters are changed and the design procedure continues until a satisfactory design is found. For a small-scale system a few design iterations may be enough to complete a satisfactory design. However, for a large-scale system, such as the space station, the dynamic model usually involves a large number of degrees of freedom and is best described by second-order dynamic equations in terms of sparse structural matrices including mass and stiffness matrices. For second-order dynamic systems, transforming to first-order form not only increases the dimension of the problem but also destroys the sparsity of the structural matrices (i.e., the mass and stiffness matrices for flexible structures). As a result, computational efficiency and physical insight are lost in the first-order form. Existing control analysis and design software may not be able to handle such a large system due to computational difficulties. For example, solving a 1000x1000 dimension Riccati equation is considered numerically impossible using today’s numerical techniques. One way to address the controller design problems for a large-scale system is to minimize the dimension of the system model by preserving the dynamic equations in second-order form and then performing model reduction. Laboratory experiments are required to verify the reduced model for robust controller designs. Recently, controller designs that directly use second-order system equations have gained attention in the literature (see Ref. 1). Their computational advantages and physical features are also illustrated in Refs. 2 and 3. Another method is to design a model-independent controller, which is insensitive to system uncertainties. The objective of this paper is to derive model-independent controllers for dynamic systems using second-order dynamic equations.

When a mass-spring dashpot system is attached to any mechanical system, including flexible space structures, the damping of the system is almost always augmented, regardless of the system size. The parameters of a teetering gondola and independent and are thus insensitive to the system uncertainties. To satisfy the system performance requirements, the parameters are adjusted using the knowledge of the system model. The more the system is known, the better the parameters of the mass-spring dashpot may be adjusted to meet the performance requirements. However, no matter what happens, the mass-spring dashpot will not destabilize the system because it is an energy-dissipative device. The question arises as to whether there are any feedback control designs using sensors and actuators that behave like the passive mass-spring dashpot. This paper is motivated by this question, and the answer is very encouraging.

A novel approach for control of flexible structures is developed using a controller that can be described by a set of second-order dynamic equations. Under certain realistic (practical) conditions, this method provides a stable system in the presence of system uncertainties. For better understanding, two major steps are involved in developing the formulation of the method. First, consider only the direct output feedback for simplicity, implying the absence of dynamics in the feedback controller. Conditions are identified in terms of the number and type of sensors and their locations to make the system asymptotically stable. Second, assume that the feedback controller contains a set of second-order dynamic equations. This case is equivalent to having a virtual flexible body (i.e., the feedback controller) that is linked side by side to the real flexible body. In other words, two sets of second-order dynamic equations are coupled to generate a closed-loop system. Design freedom increases when the dimension of the controller dynamic equations increases. Conditions are derived for the design of a stable closed-loop system with an infinite gain margin. The method takes advantage of the second-order form of equations (instead of transforming to a first-order form), which provides an easy way of discussing and obtaining the stability margin and results in a considerable computational efficiency for numerical simulations. Comparisons between the active feedback and the passive mass-spring dashpot are given through several illustrative examples.

Direct Feedback

In the analysis and design of dynamics and vibration control of flexible structures, two set of linear, constant coefficient, ordinary differential equations are frequently used:

\[ Mx + Dx + Kx = Bu \]  

(1)
where \( x \) is an \( n \times 1 \) displacement vector, and \( M, D, \) and \( K \) are mass, damping, and stiffness matrices, respectively, which generally are symmetric and sparse. The \( n \times p \) influence matrix \( B \) describes the actuator force distributions for the \( p \times 1 \) control force vector \( u \). Typically, the matrix \( M \) is positive definite, whereas \( D \) and \( K \) are positive semidefinite. In the absence of rigid-body motion, \( K \) is positive definite. Equation (2) is a measurement equation that has \( x \) as the \( m \times 1 \) measurement vector, \( H \), the \( m \times n \) acceleration influence matrix, \( B \), the \( m \times n \) velocity influence matrix, and \( Hd \) the \( m \times p \) displacement influence matrix. Note that Eq. (1) can be solved for the acceleration in terms of the displacement, velocity, and control force to obtain a new measurement equation in place of Eq. (2). However, physical insight is lost in this approach to controller design. As long as the vector \( x \) stays in the physical coordinates, the matrices \( B, H, Hd, \) and \( Hc \) are, in general, not functions of the system physical properties, including mass, damping, and stiffness.

The measurement equation [Eq. (2)] may be used either directly or indirectly for a feedback controller design. Here we will use direct feedback. Let the input vector \( u \) be

\[
\begin{aligned}
\mathbf{u} = -C\mathbf{y} = -G\mathbf{H}_{d}\mathbf{x} - G\mathbf{H}_{c}\mathbf{x} + G\mathbf{H}_{d}\mathbf{x}
\end{aligned}
\]

where \( G \) is a gain matrix to be determined. Substituting Eq. (3) into Eq. (1) yields

\[
\begin{aligned}
(M + BGH_{d})\mathbf{x} + (D + BGH_{c})\dot{\mathbf{x}} + (K + BGH_{c})\mathbf{x} = \mathbf{0}
\end{aligned}
\]

For simplicity, consider the case where \( H_{d} = H_{c} = 0 \). Assume that the number of sensors \( m \) is larger than the number of actuators \( p \). Let the actuators be located such that the row space generated by \( B^{T} \) belongs to the row space generated by \( H_{c} \). In other words, the actuators are located in such a way that the control influence matrix \( B \) can be expressed by

\[
B^{T} = C_{0}H_{c}
\]

where \( C_{0} \) is a \( p \times m \) matrix that may be obtained by \( C_{0} = B^{T}H_{c}^{-1}(H_{c}H_{c}^{T})^{-1} \). Assume that the gain matrix \( G \) is computed by

\[
G = LL^{T}C_{0}
\]

where \( L \) is a \( p \times p \) arbitrary nonzero matrix. Substituting Eq. (6) into Eq. (4) and noting the assumption that \( H_{c} = 0 \) leads to

\[
MX + (D + BLL^{T}B^{T})\dot{x} + Kx = 0
\]

The matrix \( BLL^{T}B^{T} \) is at least positive semidefinite; thus \( D + BLL^{T}B^{T} \) is at least positive semidefinite for a positive-semidefinite matrix \( D \). As a result, the closed-loop system [Eq. (7)] is stable if \( D + BLL^{T}B^{T} \) is positive semidefinite, or asymptotically stable if \( D + BLL^{T}B^{T} \) is positive definite. For the case where \( D \) is positive definite, \( D + BLL^{T}B^{T} \) is positive definite, which yields an asymptotically stable closed-loop system. This leads to a conclusion that, for a structural system with some passive damping, an output velocity feedback scheme with noncollocated velocity sensors and actuators may make the closed-loop system asymptotically stable with an infinite gain margin in the sense that the matrix \( L \) given in Eq. (6) for determination of the gain matrix \( G \) is an arbitrary (nonzero) matrix, as long as the actuators are properly located so that Eq. (6) is satisfied. Note that, for collocated sensors and actuators, \( B^{T} = H_{c} \). Without velocity measurements the system damping cannot be augmented from direct output feedback alone. However, if there are actuator dynamics involved, the system damping may be augmented by direct displacement or acceleration feedback (see Ref. 3).

Controller with Second-Order Dynamics

Assume that the controller to be designed has a set of second-order dynamic equations and measurement equations similar to the system equations, Eqs. (1) and (2):

\[
\begin{aligned}
M_{c}\ddot{x}_{c} + D_{c}\dot{x}_{c} + K_{c}x_{c} &= Bu_{c} \\
y_{c} &= H_{c}\mathbf{x}_{c} + H_{dc}\mathbf{x}
\end{aligned}
\]

Note that this is a set of equations that does not represent any physical system. In fact, this set of equations basically serves as a filter to shift the phase of measurement signals. Here \( x_{c} \) is the controller state vector of dimension \( n_{c} \), and \( M_{c}, D_{c}, \) and \( K_{c} \) are thought of as the controller mass, damping, and stiffness matrices, respectively, which generally are symmetric and positive definite to make the controller asymptotically stable. The \( n_{c} \times m \) influence matrix \( B_{c} \) describes the force distributions for the \( m \times 1 \) input force vector \( u_{c} \). Equation (9) is the controller measurement equation that has \( y_{c} \) as the measurement vector of length \( p \), \( H_{dc} \) the \( p \times n_{c} \) acceleration influence matrix, \( H_{c} \) the \( p \times n_{c} \) velocity influence matrix, and \( H_{dc} \) the \( p \times n_{c} \) displacement influence matrix. The quantities \( M_{c}, D_{c}, K_{c}, H_{dc}, \) and \( H_{c} \) are the design parameters for the controller.

Let the input vectors \( u, u_{c} \) in Eqs. (1) and (8) be

\[
\begin{aligned}
\mathbf{u} &= y_{c} = H_{ac}\mathbf{x}_{c} + H_{wc}\mathbf{x}_{c} + H_{dc}\mathbf{x} \\
u_{c} &= \mathbf{y} = H_{c}\mathbf{x}_{c} + H_{wc}\mathbf{x}_{c} + H_{dc}\mathbf{x}
\end{aligned}
\]

Substituting Eq. (10) into Eq. (1) and Eq. (11) into Eq. (8) yields

\[
\begin{aligned}
M_{c}\ddot{x}_{c} + D_{c}\dot{x}_{c} + K_{c}x_{c} &= Bu_{c} \\
y_{c} &= H_{c}\mathbf{x}_{c} + H_{dc}\mathbf{x}
\end{aligned}
\]

where

\[
\begin{aligned}
M_{c} &= \begin{bmatrix} M & \mathbf{0} \\ \mathbf{0} & M_{c} \end{bmatrix}, \\
D_{c} &= \begin{bmatrix} D_{c} \\ \mathbf{0} \end{bmatrix}, \\
K_{c} &= \begin{bmatrix} K_{c} & \mathbf{0} \\ \mathbf{0} & K_{c} \end{bmatrix}, \\
H_{c} &= \begin{bmatrix} x_{c} \\ \mathbf{0} \end{bmatrix}, \\
H_{dc} &= \begin{bmatrix} \mathbf{0} \\ x_{c} \end{bmatrix}
\end{aligned}
\]

If the design parameters, \( M_{c}, D_{c}, K_{c}, H_{dc}, \) and \( H_{c} \), are chosen so that \( M_{c}, D_{c}, \) and \( K_{c} \) are positive definite, the closed-loop system [Eq. (12)] becomes asymptotically stable.

Displacement Feedback

For better understanding of the advantage of the controller having second-order dynamic equations, consider a special case where \( H_{d} = H_{dc} = H_{c} = 0 \). To make \( K \), symmetric, it is required that

\[
BH_{dc} = H_{d}^{T}B_{c}^{T}
\]

or

\[
\begin{bmatrix} B & -H_{d}^{T} \\ -H_{d} & B_{c}^{T} \end{bmatrix} = 0
\]

For the case where the sum of the number of actuators, \( p \), and the number of sensors, \( m \), is less than the number of states, \( n \), the left matrix of Eq. (14) is a tall matrix. Unless \( B \) is in the space spanned by \( H_{d}^{T} \) or vice versa, no solutions for \( B, H_{dc} \) exist in Eq. (13). Assume that the number of sensors, \( m \), is larger than the number of actuators, \( p \). Let the actuators be located so that the row space generated by \( B^{T} \) belongs to the row space generated by \( H_{d}^{T} \); i.e., the actuators are located in such a way that the control influence matrix \( B \) can be expressed by

\[
B^{T} = Q_{h}H_{d}
\]
where \( Q_b \) is a \( p \times m \) matrix that may be obtained by \( Q_b = B^T H_d^T (H_d H_d^T)^{-1} \). Substituting Eq. (15) into Eq. (13) yields

\[
H_d^T Q_b H_{dc} - H_d^T B_T^T I
\]

(16)

Since \( H_d^T \) is a tall matrix for \( m < n \), the only possible solution is

\[
Q_b^T H_{dc} = B_T^T I
\]

(17)

For any given matrix \( H_{dc} \), this equation produces a \( B_T^T \) that makes the matrix \( K_t \) symmetric, i.e.,

\[
K_t = \begin{bmatrix} K & -B H_{dc} \\
-H_{dc}^T B_T & K_c \end{bmatrix}
\]

or

\[
K_t = \begin{bmatrix} K & -H_{dc}^T B_T \\
-B, H_d & K_c \end{bmatrix}
\]

(18)

The next question is how to choose a matrix \( H_{dc} \) that makes the closed-loop stiffness matrix \( K \), positive definite. The matrix \( K \) is positive definite, generally written as \( K > 0 \), if and only if

\[
x_t^T K_t x_t > 0
\]

(19)

for any real vector \( x_t \) except the null vector. Substituting the definition of \( K \) and \( x_t \) from Eq. (12) into Eq. (19) yields

\[
x_t^T K_t x_t = x_t^T (K - H_{dc}^T B_T B, H_d) x_t
\]

\[+ (B, H_d x - x_t) (B, H_d x - x_t) + x_t^T (K_c - I) x_t
\]

\[= x_t^T (K - B H_d H_{dc}^T B_T) x_t
\]

\[+ (H_{dc}^T B_T x - x_t) (H_{dc}^T B_T x - x_t) + x_t^T (K_c - I) x_t
\]

(20)

This equation is greater than 0 if \( B \) and \( K_c \) are chosen so that \( K - B H_d H_{dc}^T B_T \) and \( K_c - I \) are positive definite. Note that this is a sufficient condition, but it is not a necessary condition. To make Eq. (19) hold, \( K \) must be a positive-definite matrix, i.e., \( K > 0 \), and \( B \) must be chosen so that \( K - B H_d H_{dc}^T B_T \geq 0 \). This condition implies that this controller may not be able to control rigid-body motion since \( K \) in this case is only a positive semidefinite matrix, \( K \geq 0 \). To release the constraint condition, \( K - B H_d H_{dc}^T B_T > 0 \) must be increased by at least \( B H_d \)

\[c H_{dc}^T B_T \]

In other words, the system must be stiffened. This can be achieved by adding displacement feedback.

Let the input force be

\[
u = y_c - G y = H_d x - G H_d x
\]

(21)

where \( G \) is a gain matrix to be determined. Note that the velocity feedback is not considered here. Substituting Eq. (21) into the system equation, Eq. (1), the closed-loop stiffness matrix, Eq. (18), becomes

\[
K_t = \begin{bmatrix} K + B G H_d & -H_{dc}^T B_T \\
-B, H_d & K_c \end{bmatrix}
\]

(22)

If \( G \) is chosen so that

\[
G = H_d B
\]

(23)

which from Eq. (13) results in

\[B G H_d = B H_d B, H_d = H_{dc}^T B_T B, H_d
\]

The closed-loop stiffness matrix, Eq. (22), thus becomes

\[
K_t = \begin{bmatrix} K + B H_d H_{dc}^T B_T & -B H_d \\
-B_{dc} H_{dc}^T & K_c \end{bmatrix}
\]

(24)

which changes Eq. (20) to

\[
x_t^T K_t x_t = x_t^T K x_t + (B, H_d x - x_t) (B, H_d x - x_t)
\]

\[+ x_t^T (K_c - I) x_t
\]

\[= x_t^T K x_t + (H_{dc}^T B_T x - x_t) (H_{dc}^T B_T x - x_t)
\]

\[+ x_t^T (K_c - I) x_t
\]

(25)

Since \( K \) is a design parameter, the closed-loop system becomes stable as long as \( K \) is chosen larger than \( I \). An obvious choice is \( K_c = I \), where \( I \) is an identity matrix of dimension \( n \). However, this is not the best choice (this will be discussed later). To this end, it is shown that a stable closed-loop system can be designed using a feedback controller with second-order dynamic equations. The controller has an infinite gain margin in the sense that the matrices \( M_0, D_0, \) and \( K_0 \), which may be considered as the gain matrices for the controller state vector \( x \) and its derivatives, can be as large as desired without destabilizing the system as long as they are positive definite and \( K_c > I \).

A minor modification of the aforementioned design produces a better design that has physical meaning. Indeed, let

\[
B, = K_c B_c \quad \text{or} \quad B, = K_c^{-1} B_c
\]

(26)

where \( K_c \) is assumed to be positive definite, so that the solution for \( E \), exists for any given \( B \). In addition, let the gain matrix \( G \) in Eq. (23) be slightly modified as follows:

\[
G = H_d B
\]

(27)

which, with the aid of Eq. (13), results in

\[
B G H_d = B H_d B, H_d = H_{dc}^T B_T K_c B_c H_d
\]

The closed-loop stiffness matrix in this case [see Eq. (22)] thus becomes

\[
K_t = \begin{bmatrix} K + H_{dc}^T B_T K_c B, H_d & -H_{dc}^T B_T K_c \\
-K_c B_c H_d & K_c \end{bmatrix}
\]

(28)

which, in turn, changes Eq. (25) to

\[
x_t^T K_t x_t = x_t^T K x_t + (B_c H_d x - x_t) (B_c H_d x - x_t)
\]

\[= x_t^T K x_t + (H_{dc}^T B_T x - x_t) (H_{dc}^T B_T x - x_t)
\]

(29)

This equation is obviously positive if \( K \) is at least positive semidefinite (i.e., \( K \geq 0 \)). Does this design have any physical meaning? The answer is affirmative. Consider the special case where the controller is as large as the system in the sense that the number of system states \( n \) is identical to the number of controller states \( n \). Furthermore, assume that all the states are directly measurable (\( H_d = I \)) and there are \( n \) actuators collocated with the sensors (\( B = I \)). In this case, \( Q_b = I \) [Eq. (15)], \( B_c = H_{dc} = K_c \) [Eq. (17)] for \( B_c = I \), and \( G = K_c \) [Eq. (27)], which yields, from Eq. (28),

\[
K_t = \begin{bmatrix} K + K_c & -K_c \\
-K_c & K_c \end{bmatrix}
\]

(30)

For a single degree of freedom \( (n = 1) \), \( K \), represents the stiffness matrix for two springs connected in series with the spring constants \( K \) and \( K_c \).
Physical Interpretation

For a better understanding of the nature of the dynamic control designs developed here, these designs are now interpreted in physical terms. In this section, three illustrative examples will be shown, starting with a simple spring-mass system.

Example 1: Simple Spring-Mass System with a Single-Degree-of-Freedom Controller

Consider a single-degree-of-freedom spring-mass system, \( n_c = n = 1 \), with displacement measurement of the system mass. The second-order controller for this case reduces to a virtual spring-mass-dashpot system connected in series with the system mass as shown in Fig. 1.

Let the position of masses \( m \) and \( m_c \) be measured from their equilibrium states. The equations of motion for this system can be derived as follows. The force applied to the system mass \( m \) in this case is the force \( F \) transmitted through the spring \( k \). This is precisely the control force applied to the system as given in Eq. (21), with \( H_k = k_c, H_d = 1 \), and \( G = k_c \). Thus, the second-order control law is simply

\[
\mathbf{u} = F = k_c (x_c - x) \tag{31}
\]

where \( x \) is computed from

\[
m_c \ddot{x}_c + d_c \dot{x}_c + k_c x_c = k x
\]

The equation of motion that describes the closed-loop behavior is simply

\[
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & R_c & 0 & 0 \\
0 & 0 & k + k_c & -k_c \\
-k_c & -k & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\dot{x} \\
\dot{x}_c \\
x \\
x_c
\end{bmatrix} = 0
\tag{32}
\]

Equation (32) verifies Eq. (12) with \( H_k = H_d = H_k = H_d = H_c = 0 \) (i.e., no velocity and acceleration measurements) and \( K \), given in Eq. (30). Equation (32) is always stable for any \( m, k, m_c, k_c, \) and \( d \), and is asymptotically stable for any \( d > 0 \). We now consider various special cases.

Case 1

For the controller without damping, \( d_c = 0 \), the system reduces to two spring-masses connected in series. If \( k_c \) is small, the control force given in Eq. (31) is small; thus, the controller exerts little influence on the system. Mathematically, Eq. (32) becomes a set of two nearly uncoupled equations of \( x \) and \( x_c \), and obviously little change in the response of the controlled system is expected from this controller. If, however, \( k_c \) is large (i.e., the virtual spring is stiff), the relative displacement between the two masses is small. Hence, in the limit the two masses move together like a single mass \( m + m_c \), and the natural frequency of the system is approaching

\[
\omega_n = \sqrt{k / (m + m_c)} \tag{33}
\]

Case 2

For \( d > 0 \), the system is always asymptotically stable (unless \( k_c = 0 \), which, as discussed earlier, means no control). The energy flows from \( m \) to \( m_c \), and is dissipated by the damper. Again, for large \( k_c \), the system can be approximated as

\[
(m + m_c) \ddot{x} + d_c \dot{x} + k x = 0
\tag{34}
\]

Introduce the notation

\[
\frac{m + m_c}{2} \frac{m + m_c}{2}
\tag{35}
\]

Thus,

\[
\xi = \frac{1}{2} \frac{d_c}{\sqrt{k (m + m_c)}}
\]

The design variables in this case are \( d_c, m_c \). Various choices of \( d \) and \( m_c \), will result in \( \xi > 1, \xi < 1 \), or \( \xi = 1 \), which corresponds to the cases where the closed-loop system is overdamped, underdamped, or critically damped, respectively.

Case 3

For general values of \( k_c, d_c \), and \( m_c \), the design can be regarded as a virtual vibration absorber. Let the system be excited by some unknown force, \( F e^{i \omega t} \), and the displacement of the mass \( m \) be denoted by \( x = X e^{i \omega t + \phi} \). The typical objective of a vibration absorber design is to determine the values of \( k_c, d_c, \) and \( m_c \) so that the ratio

\[
\gamma = \frac{X e^{i \omega t + \phi}}{F e^{i \omega t}}
\]

is minimized over an interested range of excitation frequency \( \omega \).

Example 2: Two-Degree-of-Freedom System with a Single-Degree-of-Freedom Dynamic Controller

Next, consider a two-degree-of-freedom spring-mass system with displacement measurements of the masses \( m_1 \) and \( m_2 \) from their equilibrium positions, \( n = 2 \). First, consider the case where the controller has only one state, \( n_c = 1 \). The second-order controller in this case is simply equivalent to a virtual spring-mass-dashpot system connected in series with the two system masses as shown in Fig. 2.

It can be easily shown that the control force applied to the system is simply

\[
\mathbf{u} = F_1 \begin{bmatrix} 0 \\ k_c (x_c - x_1) \end{bmatrix}
\tag{36}
\]

where \( F_1 \) denotes the force applied to \( m_j, j = 1, 2 \); and \( x_c \) is given by

\[
m_c \ddot{x}_c + d_c \dot{x}_c + k_c x_c = k_c x_2
\]
Furthermore, the closed-loop behavior is governed by
\[
\begin{bmatrix}
    m_1 & 0 & 0 & 0 \\
    0 & m_2 & 0 & 0 \\
    0 & 0 & m_c & 0 \\
\end{bmatrix}
\begin{bmatrix}
    \dot{x}_1 \\
    \dot{x}_2 \\
    \dot{x}_c \\
\end{bmatrix}
+ \begin{bmatrix}
    k_1 + k_2 & -k_2 & 0 \\
    -k_2 & k_2 + k_c & -k_c \\
    0 & -k_c & k_c \\
\end{bmatrix}
\begin{bmatrix}
    x_1 \\
    x_2 \\
    x_c \\
\end{bmatrix}
= 0
\] (37)

which verifies Eq. (12) with \( H_s = -H_e = H_{ac} = H_c = 0 \) and with \( K_c \) given in Eq. (30). Note that this scheme requires only displacement measurement of the mass \( m_2 \).

Example 3: Two-Degree-of-Freedom System with a Two-Degree-of-Freedom Dynamic Controller
Consider the two-degree-of-freedom system given earlier with displacement measurements only, but now displacement measurement of the mass \( m_1 \) is also to be used in the controller design. The second-order controller design in this case is simply
\[
u = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \begin{bmatrix} k_{c1} & 0 \\ 0 & k_{c2} \end{bmatrix} \begin{bmatrix} x_{c1} - x_1 \\ x_{c2} - x_2 \end{bmatrix}
\] (38)

where
\[
\begin{bmatrix}
    m_{c1} & 0 \\
    0 & m_{c2} \\
\end{bmatrix}
\begin{bmatrix}
    \dot{x}_{c1} \\
    \dot{x}_{c2} \\
\end{bmatrix}
+ \begin{bmatrix}
    d_{c1} & 0 \\
    0 & d_{c2} \\
\end{bmatrix}
\begin{bmatrix}
    \dot{x}_{c1} \\
    \dot{x}_{c2} \\
\end{bmatrix}
+ \begin{bmatrix}
    k_{c1} & 0 \\
    0 & k_{c2} \\
\end{bmatrix}
\begin{bmatrix}
    x_{c1} \\
    x_{c2} \\
\end{bmatrix}
= 0
\] (39)

The closed-loop system is equivalent to a mass-spring-dashpot system shown in Fig. 3, whose behavior is governed by
\[
\begin{bmatrix}
    m_1 & 0 & 0 & 0 \\
    0 & m_2 & 0 & 0 \\
    0 & 0 & m_{c1} & 0 \\
    0 & 0 & 0 & m_{c2} \\
\end{bmatrix}
\begin{bmatrix}
    \dot{x}_1 \\
    \dot{x}_2 \\
    \dot{x}_{c1} \\
    \dot{x}_{c2} \\
\end{bmatrix}
+ \begin{bmatrix}
    k_1 + k_2 + k_c & -k_2 & -k_{c1} & 0 \\
    -k_2 & k_2 + k_c & 0 & -k_{c2} \\
    0 & -k_{c1} & k_c & 0 \\
    0 & 0 & 0 & k_c \\
\end{bmatrix}
\begin{bmatrix}
    x_1 \\
    x_2 \\
    x_{c1} \\
    x_{c2} \\
\end{bmatrix}
= 0
\] (40)

If velocity measurements are available, say, at the system mass \( m_1 \), then a dashpot element may be added in between \( m_1 \).

**Fig. 3** Two-degree-of-freedom system with a two-degree-of-freedom dynamic controller.

And \( m_{c1} \). It should be noted, however, that the controller masses, springs, and dashpots are in fact virtual elements with physical interpretations as such. For ground-based systems they may represent actual physical elements attached to the ground, but for space-based systems, they are simply controller gains in the control algorithm.

**Acceleration Feedback**

The virtual passive controller design can also be extended to acceleration feedback. Consider the system given in Eq. (1), but now the measurement vector \( y \) in Eq. (2) has only acceleration measurements: i.e., \( H_s = -H_e = H_{ac} = H_c = 0 \) in Eq. (12):
\[
M_1 \ddot{x}_1 + D_1 \dot{x}_1 + K_1 x_1 = 0
\]

where
\[
M_1 = \begin{bmatrix} M & -BH_{ac} \\ -B_c H_a & M_c \end{bmatrix}
\]
\[
D_1 = \begin{bmatrix} D & 0 \\ 0 & D_c \end{bmatrix}, \quad K_1 = \begin{bmatrix} K & 0 \\ 0 & K_c \end{bmatrix}
\]

To make \( M_1 \) symmetric, it is required that \( BH_{ac} = H_{ac}^T B_c^T \), as discussed in Eqs. (13-17). All of the discussions regarding the positive definiteness of \( K_1 \) from Eqs. (18-20) also apply to \( M_1 \).

Additional coupling in the closed-loop mass matrix \( M_1 \) can be achieved by letting the input \( u \) in Eq. (12) include direct acceleration feedback, i.e.,
\[
u = y_c - G_a y = H_{ac} \dot{x}_c - G_a y
\] (41)

which makes \( M_1 \) become
\[
M_1 = \begin{bmatrix} M + BH_{ac} & -BH_{ac} \\ -B_c H_a & M_c \end{bmatrix}
\] (42)

As before, \( M_1 \) can be made symmetric and positive definite by proper choices of \( H_{ac}, E_s, \) and \( G_a \). Let
\[
B_c = M_c \tilde{B}_c \quad \text{or} \quad B_c = M_c^{-1} \tilde{B}_c
\] (43)

where \( M_c \) is positive definite, so that the solution to \( B_c \) exists for any given \( B_c \). Let \( G \) be chosen so that
\[
G = H_{ac} \tilde{B}_c = H_{ac} M_c^{-1} B_c
\] (44)

which, with the aid of the equality \( BH_{ac} = H_{ac}^T B_c^T \), results in
\[
BGH_{ac} = BH_{ac}^T H_{ac} = H_{ac}^T B_c^T H_{ac} - H_{ac}^T B_c^T M_c \tilde{B}_c H_a
\] (45)

The closed-loop mass matrix in this case becomes
\[
M_1 = \begin{bmatrix} M + H_{ac}^T B_c^T M_c \tilde{B}_c H_a - H_{ac}^T B_c^T M_c & -M_c \tilde{B}_c H_a \\ -M_c \tilde{B}_c H_a & M_c \end{bmatrix}
\] (46)

This is a positive-definite matrix as discussed in Eq. (29) for \( K_1 \), regardless of the value of \( M \) as long as \( M \) is positive definite. The closed-loop in this case becomes
\[
\begin{bmatrix}
    M + H_{ac}^T B_c^T M_c \tilde{B}_c H_a - H_{ac}^T B_c^T M_c & -M_c \tilde{B}_c H_a \\
    -M_c \tilde{B}_c H_a & M_c \\
\end{bmatrix}
\begin{bmatrix}
    \dot{x}_1 \\
    \dot{x}_2 \\
\end{bmatrix}
+ \begin{bmatrix}
    D & 0 \\
    0 & D_c \end{bmatrix}
\begin{bmatrix}
    \dot{x}_1 \\
    \dot{x}_2 \\
\end{bmatrix}
+ \begin{bmatrix}
    K & 0 \\
    0 & K_c \end{bmatrix}
\begin{bmatrix}
    x_1 \\
    x_2 \\
\end{bmatrix}
= 0
\] (47)

Figure 4 shows a block diagram of the closed-loop system with acceleration feedback.

Let \( G_c(s) = H_{ac}[M_1 s^2 + D_c + K_c]^{-1} B_c \) be the system transfer function, \( G_c(s) = H_{ac}[M_1 s^2 + D_c + K_c]^{-1} B_c \) be the controller transfer function, and \( G_a = H_{ac} M_c^{-1} B_c \) be the direct acceleration
tion feedback gain. The acceleration measurement \( y(s) \) caused by the application of an external force \( r(s) \) can be expressed as
\[
y(s) = s^2 G_e(s) \left[ r(s) + \left( s^2 G_e(s) - G_d \right) y(s) \right]
\] (48)
or
\[
y(s) = \left[ I - s^2 G_e(s) \right]^{-1} s^2 G_e(s) r(s)
\]
The closed-loop transfer function from \( r(s) \) to \( y(s) \) is
\[
G(s) = \left[ I - s^2 G_e(s) \right]^{-1} \]

It is interesting to note that
\[
s^2 G_e(s) - G_d
\] is positive definite. The quantities \( M_e, D_e, \) and \( K_e \) are design parameters that are model independent, but they must be positive definite. The quantities \( H_{\text{ac}} \) and \( B_e \) are related by \( H_{\text{ac}} = H_{\text{ac}}M_e \) and recall that \( B_e = M_eB_c \). Equation (50) becomes
\[
s^2 G_e(s) - G_d = -H_{\text{ac}}(s^2 + D_e s + K_e^{-1})(D_e s + K_e)^{-1} B_c \] (51)
For the case where \( M_e \) is sufficiently large so that \( M_e^{-1} D_e \) and \( M_e^{-1} K_e \) may be neglected, Eq. (51) can be approximated by
\[
s^2 G_e(s) - G_d = -H_{\text{ac}}(D_e s + K_e)B_c s^{-2}
\] (52)
For the case \( K_e = 0 \) the controller becomes an integrator of the measurement acceleration. If \( H_{\text{ac}} \) is chosen to be \( B_c^T \), then
\[
s^2 G_e(s) - G_d = -B_c^T D_c B_c s^{-1}
\]
which is equivalent to a direct velocity feedback to the system.

Although it seems logical to choose a large mass matrix \( M_c \) for the controller, measurement bias and noises may prevent such a choice in practice, because integrating a bias is obviously not desirable in a control loop.

The procedure for deriving the second-order controller with acceleration feedback is identical to that for displacement feedback. Mathematically, both controllers are identical in the sense that the closed-loop mass matrix \( M_c \) for acceleration feedback can be obtained by replacing \( K \) by \( M \) in the closed-loop stiffness matrix \( \mathbf{K} \) for displacement feedback, and subscript \( q \) by \( a \). In other words, both displacement and acceleration feedback are conceptually dual. However, significant differences between both controllers appear when they are implemented, either actively or passively. This will be shown in the following example.

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**Example 4: Single-Degree-of-Freedom System with Acceleration Feedback**

Consider a single-degree-of-freedom spring-mass system with acceleration measurement of the system mass. The second-order controller for this case reduces to a virtual spring-mass dashpot connected in series with the system mass as shown in Fig. 6.

Note that the vector \( \mathbf{x}_c \) here means the relative position of \( m_c \) to the position of \( m \). In this case, \( H_a \) and \( B_c \) in Eq. (47) are chosen to be \( B_c = -H_a \mathbf{I} \). The second-order control law is
\[
u = -m_c (\ddot{x}_c + \dot{x})
\]
where \( \mathbf{x}_c \) is computed from
\[
m_c \ddot{x}_c + d_c \dot{x}_c + k_c x_c = -m_c \ddot{x}
\]
The transfer function [Eq. (49)] becomes
\[
G(s) = s^2/[\{(m s^2 + k) + m_c s^2 (d_c s + k_c) \} (m_c s^2 + d_c s + k_c)^{-1}]
\]
For large \( m_c \), \( G(s) \) reduces to
\[
G(s) \approx s^2/[\{m s^2 + d_c s + (k + k_c)\}]
\]
The system is clearly asymptotically stable. The numerator \( s^2 \) appears due to the acceleration feedback.

Comparison of Figs. 1 and 6 reveals the difference between the acceleration and displacement feedback controllers. The controller for acceleration feedback does not have a virtual ground attached to the control mass and thus cannot control the rigid-body motion.

**Conclusions**

This paper formulates a robust second-order dynamic stabilization controller design for second-order dynamic systems. The design is passive in the sense that it contains mechanisms that serve only to transfer and dissipate the energy of the system. The controller interacts with the physical system only
through spring, mass, and dashpot elements; therefore, it can be implemented actively or passively. In other words, stabilization can be accomplished either by a controller with gains interpreted as virtual mass, spring, and dashpot elements, or by actual physical masses, springs, and dashpots connected to the system.

The passive design means that the controller does not destabilize the system. As far as stability is concerned, the controller is model independent, and this is a robust design. Specifically, overall closed-loop stability is guaranteed independently of the system structural uncertainty and variations in the structural parameters. It should be emphasized that this is a robustness result with respect to structural uncertainty in the absence of measurement uncertainty and other contributing factors.

However, control performance, unlike stability robustness, is dependent on the system characteristics. Knowledge of the system model can always help improve a controller design. In this method the controller order and/or controller gains can be adjusted to meet the desired performance. Physical interpretation of the controller gains as virtual masses, springs, and dashpots provides convenient rules of thumb as to how they should be adjusted to meet a certain desired performance objective.

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References

