Single-Mode Projection Filters  
for Modal Parameter Identification for Flexible Structures

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Single-mode projection filters are developed for eigensystem parameter identification from both analytical results and test data. Explicit formulations of these projection filters are derived using the orthogonal matrices of the controllability and observability matrices in the general sense. A global minimum optimization algorithm is applied to update the filter parameters by using the interval analysis method. The updated modal parameters represent the characteristics of the test data. For illustration of this new approach, a numerical simulation for the MAST beam structure is shown by using a one-dimensional global optimization algorithm to identify modal frequencies and damping. The projection filters are practical for parallel processing implementation.

Introduction

In recent years, many researchers have investigated the problems of deriving control algorithms and state estimators for maneuvering flexible structures. The control design demands an accurate model of the system dynamics that will adequately describe the dynamic behavior of the system. System identification methods use experimental measurements to estimate dynamic properties such as natural frequencies, damping factors, mode shapes, and modal masses, which are referred to as modal parameters. Several different off-line identification of structures. Various techniques may share the same mathematical foundation via system realization theory. In order to achieve the final purpose of identification, i.e., control of flexible structures, an on-line estimation technique must be used.

For linear time-invariant systems, optimal model reduction and state estimation have been developed via optimal projection equations based on modified Riccati and Lyapunov equations. Other filtering approaches in both time and frequency domains are easy to implement and effective in rejecting uncorrelated measurement noise from simulated data. However, previous time-domain filters usually involve unacceptable computational burden for multimode systems such as large flexible structures.

In this paper, simple projection filters are developed using system realization theory, which has been used in deriving system identification methods. The development of projection filters initially was motivated by the need of an on-line technique for state estimation of linear dynamical systems. Because of modeling errors resulting from system uncertain-

ties, the projection filters naturally are required to be verified and updated from measurement data. The main objective of this paper is to present a novel approach to update the projection filters, which, in turn, yields the modal parameter identification for linear dynamical systems.

Each projection filter is formulated with a single mode only, and its explicit expression can be derived using the orthogonal matrices of the controllability and observability matrices in the general sense. The modal parameters (including modal frequency, modal damping, and mode shapes) required for formulating the projection filters are obtained initially from an analytical model in modal space. The experimental data are then passed through the projection filters to determine whether there is a discrepancy between the analytical model and the experimental testing. Each projection filter is then updated by varying the corresponding modal parameters to minimize a cost function defined by the norm of a specified error matrix. A one-dimensional global minimum optimization algorithm is applied for the filter update by using the interval analysis method, which guarantees finding the smallest value of a cost function throughout a specified closed region of modal parameters. The updated projection filters thus produce the modal parameter identification for the system.

Finally, a numerical example for the MAST truss beam structure is given to illustrate this new method.

Projection Filter Formulations

The projection filters are developed using system realization theory. A finite-dimensional, linear, time-invariant dynamic system can be represented by the state-variable equations in discrete-time form:

\[ q(k+1) = Aq(k) + Bu(k) \]  

(1)  

\[ y(k) = Cq(k) \]  

(2)

where \( q \) is an \( n \)-dimensional state vector, \( u \) an \( m \)-dimensional control or input vector, and \( y \) a \( p \)-dimensional measurement or output vector. The integer \( k \) is the sample indicator. For flexible structures, the state transition matrix \( A \) is a represen-
tation of mass, stiffness, and damping properties. The control influence matrix $B$ characterizes the locations and type of input control vector $u$. The measurement influence matrix $C$ describes the relationship between the state vector $x$ and the output measurement vector $y$, and characterizes the mode shapes of the system.

For the state-variable equations (1) and (2) with free-pulse response, the time-domain description is given by the function known as the Markov parameter

$$Y(k) = CA^k q(0)$$

or, in the case of initial state response (zero input),

$$Y(k) = CA^k$$

where $q(0)$ represents the initial conditions of the state vector and $k$ is an integer. The functions $Y(k)$ can be obtained from the measured data and used to form the $(r + 1) \times (n + 1)$ block data matrix (generalized Hankel matrix)

$$H(k - 1) = 
\begin{bmatrix}
Y(k) & Y(k + t_1) & \cdots & Y(k + t_r) \\
Y(j_1, k) & Y(j_1, k + t_1) & \cdots & Y(j_1, k + t_r) \\
Y(j_2, k) & Y(j_2, k + t_1) & \cdots & Y(j_2, k + t_r) \\
\vdots & \vdots & \ddots & \vdots \\
Y(j_r, k) & Y(j_r, k + t_1) & \cdots & Y(j_r, k + t_r)
\end{bmatrix}$$

(4)

where $j_i (i = 1, \ldots, r)$ and $t_i (i = 1, \ldots, n)$ are arbitrary integers. For the system with initial state-response measurements, simply replace $H(k - 1)$ by $H(k)$.

From Eqs. (3) and (4), it can be shown that

$$H(0) = V_s W_s^*$$

If $A$ is nonsingular, $(r + 1) p \geq n$ and $n \times m(s + 1) \geq n$, we can derive

$$V_s^* H(0) W_s^* = I_n$$

with

$$V_s^* V_r - W_s W_s^* = I_n$$

(7)

where $V_s^*$ and $W_s^*$ are the orthogonal matrices of $V_s$ and $W_s$, respectively, and $I_n$ is an identity matrix of order $n$. Instead of using the matrices $V_s^*$ and $W_s^*$ for an $n$ dimensional multimode system, simpler forms of $V_s^*$ and $W_s^*$ are derived, which represent the orthogonal matrices of the respective generalized observability and controllability matrices derived from a single-mode model only. Note that $V_s^*$ and $W_s^*$ are rectangular matrices with dimensions $2 \times (r + 1)p$ and $m(s + 1) \times 2$, respectively. The matrices $V_s^*$ and $W_s^*$, which are formulated only for the specific mode of interest from the analytical results, are used as the left and right projection filters, respectively. The Hankel matrix, $H(0)$, which is formed from measured data, will then pass through the projection filters to identify the modal parameters characterizing the measured data. If the modes of the measured data are uncoupled and distinct, and the projection filters have the same modal characteristics as the measured data, Eq. (7) yields

$$V_s^* H(0) W_s^* \approx I_2$$

(8)

where $I_2$ is a $2 \times 2$ identity matrix. The approximate equality in Eq. (8) is caused by the finite sampling time $T$ used for these digital filters. If $T$ is sufficiently small for a finite-data length the exact equality in Eq. (8) holds (see Appendix A for proof). On the other hand, if Eq. (8) does not hold it indicates that the modal parameters of the projection filters are different from those of the measured data. The projection filters should be tuned in order to match the modes of the measured data. The algorithm for the filter update is developed in the next section.

Explicit expressions of the projection filters $V_s^*$ and $W_s^*$ are derived as follows. A single-mode, continuous-time, linear, time-invariant dynamic system has the state-variable equations in modal space

$$\dot{x} = Ax + Bu$$

$$y = Cx$$

(9)

with

$$A = \begin{bmatrix} e^{-\sigma T \cos(w T)} & e^{-\sigma T \sin(w T)} \\ -e^{-\sigma T \sin(w T)} & e^{-\sigma T \cos(w T)} \end{bmatrix}$$

(10)

and

$$B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$C = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

(11)

where $\omega$ is the damped modal frequency (the imaginary part of the eigenvalue) and $-\sigma$ is the damping (the real part of the eigenvalue). Let $\omega_n$ and $\zeta$ be defined as the natural frequency and damping ratio, respectively, then $\omega = \omega_n (1 - \zeta^2)^{1/2}$ and $\sigma = \zeta \omega_n$. The corresponding single-input/single-output (SISO) discrete-time system can be represented by Eqs. (1) and (2) with

$$V = \begin{bmatrix} CA^0 \\ CA^1 \\ \vdots \\ CA^n \end{bmatrix}$$

$$W = \begin{bmatrix} b_1 V_1(s) + c_1 V_2(s) - c_2 V_1(s) + c_2 V_2(s) \\ \vdots \\ b_1 V_1(s) + b_2 V_2(s) \end{bmatrix}$$

(12)

(13)

where $b_1, b_2, c_1$, and $c_2$ are scalars. From Eq. (5), with $j_0 = t_0 = 0$, one obtains

$$V_s = \begin{bmatrix} \begin{bmatrix} CA^0 \\ CA^1 \\ \vdots \\ CA^n \end{bmatrix} - c_2 V_1(s) + c_1 V_2(s) - c_2 V_1(s) + c_2 V_2(s) \end{bmatrix}$$

(14)

$$W = \begin{bmatrix} b_1 V_1(s) + b_1 V_2(s) \\ b_1 V_1(s) + b_2 V_2(s) \end{bmatrix}$$

(15)

where

$$V_1(s) = [\ldots V_{1r} \ldots]$$

$$V_2(s) = [\ldots V_{2r} \ldots]$$

(16)

with

$$V_{1i} = -e^{-\iota \sigma T \sin(j_i \omega T)}$$

$$V_{2i} = e^{-\iota \sigma T \cos(j_i \omega T)}$$

(17)

for $i = 0, 1, 2, \ldots, r$ and

$$V_1(s) = [\ldots V_{1m} \ldots]$$

$$V_2(s) = [\ldots V_{2m} \ldots]$$

(18)

with

$$V_{3k} = -e^{-\iota \sigma T \sin(k \omega T)}$$

$$V_{4k} = e^{-\iota \sigma T \cos(k \omega T)}$$

(19)
for \( k = 0, 1, 2, \ldots, s \). Assume we choose \( j_i \) and \( t_k \) as follows:

\[
j_i - j_{i-1} = j_0, \quad i = 0, 1, 2, \ldots, \text{integer} \ [r/2]
\]

(20)

\[
t_k - t_{k-1} = t_0, \quad k = 0, 1, 2, \ldots, \text{integer} \ [s/2]
\]

(21)

Then, the projection filters, \( V^\# \) and \( W^\# \), have the following explicit expressions (see Appendix B for proof):

\[
V^\# = \begin{bmatrix} c_1 V^\# (r) + c_2 V^\# (r) \\ 1 - c_1 V^\# (r) + c_2 V^\# (r) \end{bmatrix}
\]

(22)

and

\[
W^\# = \begin{bmatrix} -b_1 V^\# (s) + b_1 V^\# (s) \\ b_1 V^\# (s) + b_1 V^\# (s) \end{bmatrix}
\]

(23)

where

\[
V^\# (r) = [ \ldots, V^\# r, \ldots ] \quad V^\# (r) = [ \ldots, V^\# r, \ldots ]
\]

(24)

\[
V^\# (r) = e^{i \eta r T} \sin(j_i \omega T)'[1/\lambda_1 (r) + 1/\lambda_2 (r)]
\]

(25)

for \( i = 0, 1, 2, \ldots, r \).

\[
V^\# (s) = [ \ldots, V^\# s, \ldots ] \quad V^\# (s) = [ \ldots, V^\# s, \ldots ]
\]

(26)

with

\[
V^\# (s) = e^{i \eta s T} \sin(t_0 \omega T)'[1/\lambda_1 (s) + 1/\lambda_2 (s)]
\]

(27)

for \( k = 0, 1, 2, \ldots, s \). The quantities \( \lambda_1 (r) \) and \( \lambda_2 (r) \) are the eigenvalues of \( V' V \), and \( \lambda_3 (s) \) and \( \lambda_4 (s) \) are the eigenvalues of \( W' W \) when \( b_1 = c_2 = 1 \) and \( b_2 = c_1 = 0 \). The quantities \( \lambda_1 (r) \), \( \lambda_2 (r) \), \( \lambda_3 (s) \), and \( \lambda_4 (s) \) are shown as follows:

\[
\lambda_1 (r) = 1 + m + Y(m), \quad \text{if} \ r \ \text{is even}
\]

(28)

\[
\lambda_2 (r) = m - Y(m), \quad \text{if} \ r \ \text{is odd}
\]

(29)

\[
\lambda_3 (s) = 1 + n - Z(n), \quad \text{if} \ s \ \text{is even}
\]

(30)

\[
\lambda_4 (s) = -n - Z(n), \quad \text{if} \ s \ \text{is odd}
\]

(31)

Note that \( V^\# \) and \( W^\# \) are rectangular matrices with dimensions \( 2 \times (r + 1) \) and \( (s + 1) \times 2 \), respectively.

Then the observability and controllability matrices \( V \) and \( W \) can be derived as follows:

\[
V = \begin{bmatrix} CA^1 \ \\
CA^2 \ \\
\vdots \ \\
CA^m \end{bmatrix}, \quad W = \begin{bmatrix} A^1 B \ \\
A^2 B \ \\
\vdots \ \\
A^m B \end{bmatrix}
\]

(35)

with

\[
B = \begin{bmatrix} \ldots, b_1 T, \ldots \end{bmatrix}, \quad C = \begin{bmatrix} \cdots, \cdots \end{bmatrix}
\]

(34)

where

Here, \( V^\# \) and \( W^\# \) are shown in Eqs. (17) and (19).

The corresponding orthogonal matrices, \( V^\# \) and \( W^\# \), are
where

\[ V^* = \begin{bmatrix} -b_1V_{x1} + b_{11}V_{x1}^{*} & \frac{1}{2}mb_x^2 + mb_{11} \\ \vdots & \vdots \\ -b_{2m}V_{x1} + b_{21}V_{x1}^{*} & \frac{1}{2}mb_x^2 + mb_{21} \end{bmatrix} \]

and

\[ W^* = \begin{bmatrix} -b_1V_{x1} + b_{11}V_{x1}^{*} & \frac{1}{2}mb_x^2 + mb_{11} \\ \vdots & \vdots \\ -b_{2m}V_{x1} + b_{21}V_{x1}^{*} & \frac{1}{2}mb_x^2 + mb_{21} \end{bmatrix} \]

The algorithm is summarized as follows:

**Initial Step**

The algorithm starts with an initial interval \( I \). This interval is subdivided equally into subintervals, which are stored in a list \( L_0 \). A list \( L_1 \) (initially empty) consists of intervals for which the width is smaller than a specified value \( w_1 \), and the corresponding width of \( J \) is smaller than a specified value \( w_2 \). Let \( \tilde{x} \) denote a feasible approximation to the global minimum. If the feasible point is not given, the upper limit of the cost function is set to \( J = \infty \) with \( \tilde{x} \) indefinite. Let \( \{ j_x, j_y \} \) and \( \{ j'_x, j'_y \} \) denote the interval resulting from evaluating \( J, J' \), and \( J'' \) in interval arithmetic using the argument \( X \), respectively. That is,

\[ J(X) = \{ j_x, j_y \}, \quad J'(X) = \{ j'_x, j'_y \}, \quad J''(X) = \{ j''_x, j''_y \} \]

Then, use the interval analysis to find the corresponding \( J, J' \), and \( J'' \) for all the subintervals in \( L_0 \).

**Main Steps**

1. If the list \( L_0 \) is empty, go to step 11. Otherwise, find the subinterval \( X \) in \( L_0 \) for which the length end-point of \( J(X) \), i.e., \( J_L \), is smallest.
2. If \( x \subset X \), set \( x = \tilde{x} \). Otherwise, set \( x = m(X) = \text{midpoint of } X \). If \( J_{j_x} > 0 \) and \( J_{j_y} > 0 \), it implies that \( J'(x) > 0 \) for any point inside the interval in \( L_0 \) exceeds the upper limit \( J \). Then \( L_0 \) is empty, go to step 11.
3. Concavity check—if \( f_{j''_x} < 0 \), \( J \) is concave in \( X \) and cannot have a minimum in the interior of \( X \). Then, \( X \) is deleted, go to step 1.
4. Monotonicity check—if \( f_{j'_x} < 0 \) or \( j'_y > 0 \), the gradient of \( J \) is strictly positive or strictly negative over \( X \). Then, \( X \) is deleted, go to step 1.
5. Gaussian elimination—Denote \( E = J - J(x) \), \( A = [J'(x)]^2 + 2Ej'_x \). If \( f_{j''_x} > 0 \) and \( A > 0 \), it implies that \( J'(y) > 0 \) for any \( y \in X \). Then, \( X \) is deleted, go to step 1. Note that this is true only for \( f_{j''_x} > 0 \), which is not indicated in Ref. 5.
6. Interval Newton method—if \( f_{j''_x} > 0 \), denote \( S' = x - J'(x)/J''(x) \) and \( S = \text{intersection of } S' \) and \( X \). Otherwise, denote \( S = S_1 \cup S_2 \). Here \( S_1 \) and \( S_2 \) are defined as follows: Denote

\[ c = x - J'(x)/j''_x, \quad d = x - J'(x)/j''_y \]

and

\[ S_1 = [x_L, x], \quad S_2 = [x, x_R], \quad S_3 = [x_L, x_R] \]

If \( J'(x) \geq 0 \),

\[ S_1 = [x_L, x], \quad S_2 = [x, x_R], \quad S_3 = [x_L, x_R] \]

When \( f_{j''_x} > 0 \) and \( d > x_L \),

\[ S_1 = [x_L, x], \quad S_2 = [x, x_R], \quad S_3 = [x_L, x_R] \]

When \( f_{j''_x} < 0 \) and \( c < x_L \),

\[ S_1 = [x_L, x], \quad S_2 = [x, x_R], \quad S_3 = [x_L, x_R] \]

When \( f_{j''_x} > 0 \) and \( d > x_L \),

\[ S_1 = [x_L, x], \quad S_2 = [x, x_R], \quad S_3 = [x_L, x_R] \]

When \( f_{j''_x} < 0 \) and \( c < x_L \),

\[ S_1 = [x_L, x], \quad S_2 = [x, x_R], \quad S_3 = [x_L, x_R] \]
If \( J'(x) < 0 \),

\[
S_1 = [x_l, c], \quad \text{when } J''(x) < 0 \text{ and } c \geq x_L,
\]

\[
S_2 = [d, x_R], \quad \text{when } J''(x) = 0 \text{ or } c < x_L,
\]

\[
\text{empty,} \quad \text{when } J''(x) > 0 \text{ or } d > x_R,
\]

7) If \( S \) is empty, go to step 1. If \( S = X \), then split \( X \) in half.
8) For each newly generated subinterval, \( X = S_1, S_2, \) or \( S \), repeat steps 3 and 4.
9) Update \( J \). For each new subinterval \( \tilde{X} \), denote by \( w[\tilde{X}] \) width of \( \tilde{X} \), \( 4 = \text{mid}[X] = \text{midpoint of } X \).

\[
\tilde{X} = [\tilde{x}_L, \tilde{x}_R] \quad \text{and} \quad J(\tilde{X}) = [\tilde{J}_L, \tilde{J}_R]
\]

If \( J(\tilde{x}) < J \) simply replace \( J \) by \( J(\tilde{x}) \) or conduct a line search to reduce \( J \) as follows:

a) If \( J'(\tilde{x}) > 0 \), denote \( \tilde{x}_1 = \tilde{x}_L \). Otherwise, denote \( \tilde{x}_1 = \tilde{x}_R \).

b) Denote \( \tilde{x}_2 = (\tilde{x}_0 + \tilde{x}_1)/2 \). If \( J(\tilde{x}_2) \geq \max\{J(\tilde{x}_0), J(\tilde{x}_1)\} \), go to step e.

c) If \( J(\tilde{x}_0) \leq J(\tilde{x}_1) \), replace 4, by \( \tilde{x}_0 \). Otherwise replace \( \tilde{x}_0 \) by \( \tilde{x}_2 \).

d) If \( |\tilde{x}_1 - \tilde{x}_2| > w[\tilde{X}] \), go to step b.

e) Set \( J = \min\{J(\tilde{x}), J(\tilde{x}_0), J(\tilde{x}_1)\} \) and set \( \tilde{x} \) to the corresponding argument of \( J \).

Fig. 1 MAST truss beam structure.

Table 1 Percentage error of estimated modal frequency

<table>
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<tr>
<th>Damping ratio, %</th>
<th>0</th>
<th>0.3</th>
<th>2</th>
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<td><strong>Frequency, Hz</strong></td>
<td><strong>Noise level, %</strong></td>
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<td>5</td>
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<tr>
<td>1.4222</td>
<td>0.00³</td>
<td>0.01</td>
<td>0.02</td>
</tr>
<tr>
<td>1.4222</td>
<td>0.02</td>
<td>0.06</td>
<td>0.16</td>
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<td>8.5545</td>
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<td>9.4954</td>
<td>0.29</td>
<td>0.30</td>
<td>0.30</td>
</tr>
</tbody>
</table>

³After numerical truncation
two measurements on each bay for each direction.

Fig. 3 Mode shape of the second mode of the MAST truss beam structure. The solid bars indicate the displacements in the \( x \) direction. The empty bars indicate the displacements in the \( y \) direction. There are two measurements on each bay for each direction.

Fig. 4 One of the simulated free-impulse response data \( Y(k) \) for the MAST truss beam structure with 10% measurement noise and zero damping.

<table>
<thead>
<tr>
<th>Frequency, Hz</th>
<th>0</th>
<th>0.3</th>
<th>2</th>
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<tbody>
<tr>
<td>Noise level, %</td>
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</tr>
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<tr>
<td>0.50%</td>
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<td>1.00%</td>
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<td>1.4222</td>
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<table>
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<th>Table 2 Percentage error of estimated modal damping</th>
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<tr>
<td>Frequency, Hz</td>
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Estimated modal damping is 0% After numerical truncation.

Concluding Remarks

Projection filters are derived for possible application to the state estimation of linear dynamic systems. An approach to update the projection filter through the use of measurement data is developed to identify frequency and damping of the system. Numerical results for the MAST truss beam structure demonstrate that repeated modal frequencies can be identified within 1% error for 10% measurement noise.

There are two characteristics of the projection filters. First, each projection filter is developed based on a single-mode subsystem, which identifies only one modal frequency and one modal damping within a specified region. Second, for an \( n \)-mode structure (based on the analytical model), \( n \) single-mode projection filters can be implemented for parallel processing to reduce the computational burden. Application of the projection filters to the state estimation for linear dynamical systems is currently under investigation.

Appendix A

For a finite data length and decoupled distinct \( M \)-mode system without a measurement noise, it can be verified that

\[
\lim_{T \to 0} V^T H(0) W^T = I_2
\] (A1)
where $V_{j}^{*}$ and $W_{j}^{*}$ are the projection filters for the jth mode. On the other hand,

$$V_{j}^{*}VV_{j}^{*} = I,$$  \hspace{1cm} (A2)

where $V_{j}$ and $W_{j}$ are the respective generalized observability and controllability matrices for jth-mode subsystem.

From Eqs. (5), (6), and (A2), it is shown that

$$V_{j}^{*}H(0)W_{j}^{*} = V_{j}^{*}V_{j}W_{j}^{*} = \sum_{k=1}^{M} V_{j}^{*}V_{k}W_{j}^{*}$$

$$= I_{2} = \sum_{k=1}^{M} V_{j}^{*}V_{k}W_{j}^{*},$$  \hspace{1cm} (A3)

Now, recalling Eqs. (14) and (22) and using the subscripts $j$ and $k$ to indicate the corresponding values for the jth and kth modes, respectively, one obtains

$$V_{j}^{*}V_{k} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$  \hspace{1cm} (A4)

where

$$a_{11} = (c_{2j}c_{2k}V_{j}^{*}V_{k} + c_{1j}c_{1k}V_{j}^{*}V_{k} + c_{1j}c_{2k}V_{j}^{*}V_{2k} + c_{2j}c_{1k}V_{j}^{*}V_{2k})(c_{2j}^{2} + c_{2k}^{2})$$

Combining Eqs. (16), (17), (24), (25), and (28–30), yields

$$V_{j}^{*}V_{1k} = \sum_{r=0}^{r} e^{i(j \omega_{k} T)} \sin(j \omega_{k} T) = \frac{1}{2} \sin(j \omega_{k} T)$$

if $j$ is odd, then

$$\hat{\alpha}_{j} = (r + 1)/2 + \sum_{r=0}^{r} \frac{1}{2} \cos((r_{i} - 2j) \omega_{k} T) = (r + 1)/2 + Y(m)$$  \hspace{1cm} (A6)

Let $j_{1}$ and $t_{k}$ be chosen such that

$$j_{1} = h_{1}, \quad i = 0, 1, 2, ..., r$$

$$t_{k} = h_{2}, \quad k = 0, 1, 2, ..., S$$  \hspace{1cm} (A8)

where $h_{1}$ and $h_{2}$ are positive integers. If $h_{1}$, $h_{2}$, and the sampling time $T$ are chosen so that $h_{1}T$ and $h_{2}T$ are small, then

$$Y(m) = \sum_{r=0}^{(r-1)/2} \cos((r - 2i) h_{1} \omega_{k} T) = \sum_{r=0}^{(r-1)/2} \cos(2i h_{1} \omega_{k} T)$$

$$= \cos(\omega_{h_{1}} T) \left[ \sum_{r=0}^{(r-1)/2} \cos(2 \omega_{h_{1}} T) \right]$$

$$= \cos(\omega_{h_{1}} T) \left[ \sum_{r=0}^{(r-1)/2} \sin(2 \omega_{h_{1}} T) \right]$$

$$\approx \cos(\omega_{h_{1}} T) \int_{0}^{(r-1)/2} \cos(2 \omega_{h_{1}} T) \, dt$$

$$= \cos(\omega_{h_{1}} T) \int_{0}^{(r-1)/2} \sin(2 \omega_{h_{1}} T) \, dt$$

$$= \cos(\omega_{h_{1}} T) \sin[(r - 1) \omega_{h_{1}} T] / (2 \omega_{h_{1}} T)$$

$$+ \sin(\omega_{h_{1}} T) \{ \cos[(r - 1) \omega_{h_{1}} T] - 1) / (2 \omega_{h_{1}} T) \}$$

$$= \sin(\omega_{h_{1}} T) / (2 \omega_{h_{1}} T)$$  \hspace{1cm} (A9)

$$\left| Y(m) \right| \leq 1/(\omega_{h_{1}} T) < \infty$$  \hspace{1cm} (A10)

Similarly, from Eq. (A5), it is shown that

$$F_{i} = \sum_{i=0}^{r} \{ \sin(j \omega_{i} T) \sin(j \omega_{k} T) \}$$

$$+ \sin[(j \omega_{i} T) \sin[(j - j) \omega_{k} T] \sin(j \omega_{k} T) / (2 \omega_{k} T)$$

$$+ \sum_{j=0}^{j} e^{i(j \omega_{k} T)} \sin(j \omega_{k} T) / (2 \omega_{k} T)$$

Combining Eqs. (16), (17), (24), (25), and (28–30), yields

$$V_{j}^{*}V_{1k} = \sum_{r=0}^{r} e^{i(j \omega_{k} T)} \sin(j \omega_{k} T)$$

$$= \sin(j \omega_{k} T) \sin[(j_{i} = 2j) \omega_{k} T] / 2 \omega_{k} T$$

if $j$ is even, then

$$\hat{\alpha}_{j} = (r + 1)/2 + \sum_{r=0}^{r} \cos((r_{i} - 2j) \omega_{k} T) = (r + 1)/2 + Y(m)$$  \hspace{1cm} (A6)

This is also true if $r$ is even. Similarly, it can be proved that

$$\lim_{T \to 0} \frac{F_{i}}{(2 \omega_{k} T)} = 0$$  \hspace{1cm} (A12)

Substitution of Eqs. (A13) and (A14) into Eq. (A5) yields

$$\lim_{T \to 0} \frac{V_{j}^{*}V_{1k}}{T} = 0$$  \hspace{1cm} (A15)

For Eq. (A4), similar procedures can be used to verify

$$\lim_{T \to 0} \frac{V_{j}^{*}V_{2k}}{T} = \lim_{T \to 0} \frac{V_{j}^{*}V_{2k}}{T} = 0$$  \hspace{1cm} (A16)

and

$$\lim_{T \to 0} \frac{V_{j}^{*}V_{k}}{T} = \lim_{T \to 0} \frac{V_{j}^{*}V_{k}}{T} = 0$$  \hspace{1cm} (A17)

where 0 is a 2 x 2 zero matrix. Substitution of Eq. (A17) into Eq. (A3) derives Eq. (A1). Note that a better approximation of Eq. (8) can be achieved if smaller $T$ are used for a finite data length.
Appendix B

For the controllability matrix \( W \) and the observability matrix \( V \) shown in Eqs. (14) and (15), the corresponding orthogonal matrices, \( W^\# \) and \( V^\# \), can be derived as follows. First, observe that

\[
V^\# V = W W^\# - I_2 \tag{B1}
\]

From Eqs. (16), (17), (24), and (25), it is shown that

\[
V^\#(r)V_1(r) = \sum_{r=1}^{r/2} \{ \sin^2(j\omega T) + \sin(j\omega T) \sin[(j, j)\omega T] \}
\]

If \( r \) is even, combining Eqs. (20) and (28–30) leads to

\[
\sum_{r=0}^{r/2-1} \{ \sin^2(j\omega T) + \sin(j\omega T) \sin[(j, j)\omega T] \}
\]

\[
= \sum_{r=0}^{r/2-1} [\sin(j\omega T) + \sin((j, j)\omega T)]^2 + 2 \sin^2((j, j)\omega T)
\]

\[
= \sum_{r=0}^{r/2-1} \{ 4 \cos^2((j, j)/2) \omega T) \sin^2(j\omega T/2) \}
\]

\[
+ 2 \sin^2(j\omega T/2) \left( 1 + \sum_{r=0}^{r/2-1} 2 \cos^2((j, j)\omega T) \right)
\]

\[
= 2 \sin^2(j\omega T/2) \left[ 1 + \sum_{r=0}^{r/2-1} 1 + \cos((j, j)\omega T) \right]
\]

\[
= 2 \sin^2(j\omega T/2) \lambda_2(r) \tag{B3}
\]

If \( r \) is odd, combining Eqs. (20) and (28–30) leads to

\[
\sum_{r=0}^{r/2-1} \{ \sin^2(j\omega T) - \sin(j\omega T) \sin[(j, j)\omega T] \}
\]

\[
= \sum_{r=0}^{r/2-1} \{ \sin(j\omega T) - \sin((j, j)\omega T) \}
\]

\[
= \sum_{r=0}^{r/2-1} \{ 4 \cos^2(j\omega T/2) \sin^2((j, j)\omega T) \}
\]

\[
= 2 \cos^2(j\omega T/2) \sum_{r=0}^{r/2-1} [1 - \cos((j, j)\omega T) \}
\]

\[
= 2 \cos^2(j\omega T/2) \lambda_2(r) \tag{B4}
\]

If \( r \) is even, combining Eqs. (20) and (28–30) leads to

\[
\sum_{r=0}^{r/2-1} \{ \sin^2(j\omega T) - \sin(j\omega T) \sin[(j, j)\omega T] \}
\]

\[
= \sum_{r=0}^{r/2-1} \{ \sin(j\omega T) - \sin((j, j)\omega T) \}
\]

\[
= \sum_{r=0}^{r/2-1} \{ 4 \cos^2(j\omega T/2) \sin^2((j, j)\omega T) \}
\]

\[
= 2 \cos^2(j\omega T/2) \sum_{r=0}^{r/2-1} [1 - \cos((j, j)\omega T) \}
\]

\[
= 2 \cos^2(j\omega T/2) \lambda_2(r) \tag{B5}
\]

\[
\sum_{r=0}^{r/2-1} \{ \sin^2(j\omega T) - \sin(j\omega T) \sin[(j, j)\omega T] \}
\]

\[
= \sum_{r=0}^{r/2-1} \{ \sin(j\omega T) - \sin((j, j)\omega T) \}
\]

\[
= \sum_{r=0}^{r/2-1} \{ 4 \cos^2(j\omega T/2) \sin^2((j, j)\omega T) \}
\]

\[
= 2 \cos^2(j\omega T/2) \sum_{r=0}^{r/2-1} [1 - \cos((j, j)\omega T) \}
\]

\[
= 2 \cos^2(j\omega T/2) \lambda_2(r) \tag{B6}
\]

Substitution of Eqs. (B3–B6) into Eq. (B2) yields

\[
V^\#(r)V_1(r) = [2 \sin^2(j\omega T/2) \lambda_2(r)]/2 \lambda_2(r)
\]

\[
= 1 \tag{B7}
\]

Similarly,

\[
V^\#(r)V_2(r) = 1 \tag{B8}
\]

Next, from Eqs. (16), (17), (24), and (25), it is shown that

\[
V^\#(r)V_3(r) = \sum_{r=-2}^{r=2} \{ \sin(j\omega T) \cos(j\omega T)
\]

\[
+ \sin[(j, j)\omega T)] \cos(j\omega T)]/2 \lambda_2(r)
\]

\[
+ \sum_{r=-2}^{r=2} \{ \sin((j, j)\omega T) \cos(j\omega T)
\]

\[
+ \sin((j, j)\omega T) \cos((j, j)\omega T)]/2 \lambda_2(r) \tag{B9}
\]

If \( r \) is even, combining Eqs. (20) and (28–30) leads to

\[
\sum_{r=0}^{r/2-1} \{ \sin(j\omega T) \cos(j\omega T) + \sin((j, j)\omega T) \cos(j\omega T) \}
\]

\[
= \sum_{r=0}^{r/2-1} \{ \sin(j\omega T) + \sin((j, j)\omega T) \}
\]

\[
= \sum_{r=0}^{r/2-1} \{ 4 \cos^2((j, j)/2) \omega T) \sin^2((j, j)\omega T/2) \}
\]

\[
+ 2 \sin^2((j, j)\omega T/2) \left( 1 + \sum_{r=0}^{r/2-1} 2 \cos^2((j, j)\omega T) \right)
\]

\[
= 2 \sin^2((j, j)\omega T/2) \left[ 1 + \sum_{r=0}^{r/2-1} 1 + \cos((j, j)\omega T) \right]
\]

\[
= 2 \sin^2((j, j)\omega T/2) \lambda_2(r) \tag{B10}
\]

If \( r \) is odd, combining Eqs. (20) and (28–30) leads to

\[
\sum_{r=0}^{r/2-1} \{ \sin(j\omega T) \cos(j\omega T) + \sin((j, j)\omega T) \cos(j\omega T) \}
\]

\[
= \sum_{r=0}^{r/2-1} \{ \sin(j\omega T) - \sin((j, j)\omega T) \}
\]

\[
= \sum_{r=0}^{r/2-1} \{ 4 \cos^2((j, j)/2) \omega T) \sin^2((j, j)\omega T/2) \}
\]

\[
+ 2 \sin^2((j, j)\omega T/2) \left( 1 + \sum_{r=0}^{r/2-1} 1 + \cos((j, j)\omega T) \right)
\]

\[
= 2 \sin^2((j, j)\omega T/2) \lambda_2(r) \tag{B11}
\]
Substitution of Eqs. (B10) and (B11) into Eq. (B9) yields

\[
V_a^*(r)V_2(r) = -\frac{[\sin(j_1\omega r)\lambda_1(r)]}{2\lambda_1(r)} + \frac{[\sin(j_2\omega r)\lambda_2(r)]}{2\lambda_2(r)} = 0
\]  

(B12)

This is also true if \( r \) is odd.

Similarly, it can be proved that

\[
V_2^*(r)V_1(r) = 0
\]  

(B13)

Observation of Eqs. (14), (22), (B7), (B8), (B12), and (B13) leads to

\[
V^*V = I_2
\]  

(B14)

Similar procedures can be used to verify

\[
WW^* = I,
\]  

(B15)

Note that, if \( \sigma = 0 \), from Eqs. (14), (15), (22), and (23) it can be proved that

\[
(VV^*)^T = VV^*, \quad (WW^*)^T = WW^*
\]  

(B16)

From Eqs. (B14–B16), it is shown that \( V^* \) and \( W^* \) are the pseudoinverse matrices of \( V \) and \( W \), respectively. However, for \( \sigma \neq 0 \), \( V^* \) and \( W^* \) are not the pseudoinverse matrices. This raises doubts about the uniqueness of the modal parameters identified. The effect of these nonunique parameters needs further study in the future.

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References


