Natural Modes of Oscillation of Rotating Flexible Structures about Nontrivial Equilibrium

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This paper is concerned with the vibrational characteristics of rotating flexible structures in the neighborhood of steady rotation. It is assumed that the structure undergoes deformations during the steady rotation, so that the equilibrium is nontrivial. The object of the paper is to formulate the eigenvalue problem associated with small oscillations of the structure about the nontrivial equilibrium and to develop efficient methods to compute the system natural frequencies and modal vectors. A numerical example showing the procedure for the calculation of spacecraft modes is presented.

Nomenclature

\( a, b, c \) = orbital axes
\( x, y, z \) = spacecraft body axes
\( \theta_i(t) \) = angular displacements
\( \gamma_i(t), w_i(t) \) = elastic displacements
\( L_i(t) \) = Lagrangian functional
\( \dot{L}_i(x_i(t)) \) = Lagrangian density function
\( T_{0i}, T_{1i}, T_{2i} \) = kinetic energy components
\( V_i \) = potential energy
\( \phi_i(x_i), \psi_i(x_i) \) = admissible functions
\( \{ q(t) \} \) = \( n \)-dimensional configuration vector
\( [m] \) = \( n \times n \) inertia matrix
\( [g] \) = \( n \times n \) gyroscopic matrix
\( [k] \) = \( n \times n \) stiffness matrix
\( [x(t)] \) = \( 2n \times 1 \)-dimensional state vector
\( [I] \) = \( 2n \times 2n \) matrix defined by Eqs. (23)
\( [G] \) = \( 2n \times 2n \) matrix defined by Eqs. (23)
\( \omega_n \) = natural frequency of spacecraft
\( [K], [K'] \) = \( 2n \times 2n \) matrices defined by Eqs. (27) and (30), respectively
\( A_{0i}, B_{0i}, C_0 \) = moments of inertia of rigid hub about \( x, y, z \), respectively
\( \rho_i \) = mass density of rod \( i \)
\( m_i \) = tip mass of rod \( i \)
\( l_i \) = length of rod \( i \)
\( EI_{0i}, EI_{1i} \) = flexural stiffnesses of rod \( i \)
\( h_{m}, h_{m}, h_{m} \) = distances from mass center to point of attachment of rod \( i \)
\( \Omega \) = spacecraft spin velocity and orbital velocity
\( \alpha \) = angle defining orientation of radial booms
\( \beta \) = angle defining orientation of damper booms

I. Introduction

A problem of current interest is the dynamic behavior of flexible spacecraft rotating in space. In particular, the interest lies in the natural modes of oscillation of the complete vehicle, where natural modes are to be interpreted as including both the system natural frequencies and mode shapes. The concept of a modal vector that includes both the angular perturbations relative to a uniformly rotating frame and the elastic displacements relative to a given set of body axes is relatively new. References 1 and 2 introduced the concept and applied it to the development of a modal analysis for rotating structures.

In many cases, the oscillation takes place about an equilibrium in which the structure is deformed and the angular coordinates are different from zero; we shall refer to such equilibrium as nontrivial. Although this problem can be very important, it has received very little attention. Yet omission of this effect can lead to grossly erroneous results, particularly when the nontrivial equilibrium consists of large displacements. To illustrate the idea, let us consider an L-shaped bar fixed at one end and free at the other end and spinning uniformly with the angular velocity \( \Omega \) about an axis through the fixed end. For simplicity, let us assume that the bar segment with the fixed end is normal to the spin axis and the segment with the free end is parallel to the spin axis. The centrifugal forces will cause the bar to undergo deformations proportional to \( \Omega^2 \). The deformation pattern depends on space but not on time and it represents the state of nontrivial equilibrium. For large \( \Omega \) or small bending stiffness, the equilibrium problem may require the solution of nonlinear differential equations. If the system is perturbed slightly and the equilibrium is stable, then the system will undergo small oscillations about the nontrivial deformed equilibrium and not about the trivial undeformed configuration. Note that the undeformed configuration is not an equilibrium state for this rotating system. This leads to a very important conclusion, namely, that the system can undergo large total displacements but the oscillations can remain small. It is easy to recognize that serious errors can occur if the total displacements are mistaken for oscillations, particularly for large nontrivial equilibrium.

Problems involving nontrivial equilibrium can arise in the case of gravity-gradient of spin-stabilized spacecraft with very
flexible appendages that are not aligned with the spacecraft principal axes. Addressing himself to this problem, Flatey\(^4\) obtained the nonlinear nontrivial equilibrium configuration of the Radio Astronomy Explorer (RAE) satellite by means of an analog computer. In deriving stability criteria for the RAE/B satellite, Meirovitch\(^5\) obtained the nonlinear deformed equilibrium as a by-product, thus confirming the results of Ref. 3. Also inspired by the RAE satellite, Newton and Farrell\(^3\) presented a method for the evaluation of the natural frequencies of a flexible gravity-gradient stabilized satellite oscillating about the deformed equilibrium. As generalized coordinates. Ref. 5 considered complete deformation patterns and angular displacements of the satellite. This approach tends to limit the number of degrees of freedom of the simulation. Moreover, one must guess in advance configuration patterns, which may not be possible for complicated structures. Hence, a more general approach is desirable.

This paper is concerned with the formulation of the eigenvalue problem for oscillation about nontrivial equilibrium, and with the development of efficient methods for the solution of the eigenvalue problem for high-order systems. The problem can be divided into four parts: a) derivation of the nontrivial equilibrium; b) derivation of the variational equations of motion; c) system discretization and derivation of the eigenvalue problem; d) solution of the eigenvalue problem. The paper concentrates on the last three aspects of the problem. The derivation of the nontrivial equilibrium was presented in Ref. 4. It should be pointed out that, for modal analysis, one must have at his disposal not only the eigenvalues but also the eigenvectors of the system.

As an illustration, the eigenvalue problem for a 30 x 30 matrix is solved. The mathematical model represents a 15 degree-of-freedom simulation of the RAE/B satellite.

II. Problem Formulation

We shall be concerned with the dynamic characteristics of a spacecraft consisting of a main rigid body with \(n\) flexible appendages. The appendages can be regarded as one-dimensional continua and, in addition to their own distributed mass, they possess concentrated masses at their tips. The mass center of the spacecraft is assumed to move in a circular orbit about a center of force fixed in an inertial space. Then denoting by \(abc\) a system of orital axes and identifying a \(xyz\) domain and by \(Vi(x, t)\) the flexural displacements along the orthogonal axes \(y_i\) and \(z_i\), respectively, we can write the Lagrangian in the general functional form

\[
L(t) = L_0(t) + \sum_{i=1}^{n} \left[ \int_{0}^{\epsilon_i} L_i(x_i(t)) dx_i + L_i(t, i, t) \right]
\]

where

\[
L_0(t) = L_0 [\theta_j(t), \dot{\theta}_j(t)] \quad j = 1, 2, 3
\]

\[
L_i(x_i(t)) = L_i [\theta_j(t), \dot{\theta}_j(t), v_i(x_i(t)), \dot{v}_i(x_i(t))]
\]

and

\[
L_i(t, i, t) = L_i [\theta_j(t), \dot{\theta}_j(t), v_i(x_i(t)), \dot{v}_i(x_i(t)), \omega_i(x_i(t)), \omega_i(x_i(t)), \dot{\omega}_i(x_i(t)), \dot{\omega}_i(x_i(t))]
\]

in which \(L_0\) is the Lagrangian corresponding to the system in undeformed state, \(L_i\) the Lagrangian density associated with any point of the elastic member \(i\), and \(L_0\) the Lagrangian corresponding to the tip mass at \(x_i = \epsilon_i\), where \(\epsilon_i\) represents the length of the member \(i\). Note that dots designate differentiations with respect to the time \(t\) and primes designate differentiations with respect to the spatial coordinate \(x_i\).

III. Nontrivial Equilibrium

Equations (1) and (2) can be used in conjunction with Hamilton's principle to derive Lagrange's equations of motion about the trivial solution, namely, the solution for which all the rotational and elastic displacements and velocities are zero. These equations, however, are not very suitable when the system admits a nontrivial solution as an equilibrium state, where a nontrivial equilibrium is defined as a set of variables \(\theta_{ij}, \dot{\theta}_{ij}, \omega_i\) constant in time and satisfying Lagrange's equations. Indeed, it is shown in Ref. 6 that the nontrivial equilibrium is a solution of the equations

\[
\frac{\partial L}{\partial \theta_j} = 0 \quad j = 1, 2, 3
\]

and

\[
\frac{\partial L_i}{\partial v_i} - \frac{\partial}{\partial \theta_j} \left[ \frac{\partial L_i}{\partial \dot{v}_i} \right] + \frac{\partial^2 L_i}{\partial \dot{v}_i \partial \dot{v}_i} \dot{v}_i = 0, \quad 0 < x_i < \epsilon_i
\]

\[
\left[ \frac{\partial L_i}{\partial \dot{v}_i} - \frac{\partial}{\partial \theta_j} \left[ \frac{\partial L_i}{\partial \dot{v}_i} \right] + \frac{\partial L_i}{\partial \dot{v}_i} \right] \dot{\omega}_i = 0, \quad 0 < x_i < \epsilon_i
\]

\[
\left[ \frac{\partial L_i}{\partial \dot{v}_i} - \frac{\partial}{\partial \theta_j} \left[ \frac{\partial L_i}{\partial \dot{v}_i} \right] + \frac{\partial L_i}{\partial \dot{v}_i} \right] \dot{\omega}_i = 0, \quad 0 < x_i < \epsilon_i
\]

as well as a set of equations similar to Eq. (4) for \(\omega_i\). We shall denote the solutions of Eqs. (3) and (4), together with the set of equations for \(\omega_i\) by \(\theta_{ij}, \dot{\theta}_{ij}, \omega_i(x_i(t))\), \(\omega_i(x_i(t))\), where the first are constant and the latter are functions of the spatial variables alone.

IV. Perturbations about Equilibrium: Variational Equations of Motion

The interest lies in the oscillatory characteristics of the system about the nontrivial solution \(\theta_{ij}, \dot{\theta}_{ij}, \omega_i(x_i(t))\). In particular, we wish to calculate the natural frequencies and the natural modes of oscillation of the entire spacecraft about the nontrivial equilibrium. To this end, we must first obtain Lagrange's equations of motion for small oscillations about that equilibrium. Denoting these oscillations by \(\theta_{ij}(t), v_i(x_i(t)), \dot{v}_i(x_i(t))\) for the equations of motion of the spacecraft regarding the nontrivial solution, we can write

\[
\text{as well as a set of equations similar to Eq. (4) for } \dot{v}_i(x_i(t))\]

\[
\text{for the equations of motion of the spacecraft regarding the nontrivial solution, we can write}
\]

\[
\text{as well as a set of equations similar to Eq. (4) for } \dot{v}_i(x_i(t))\]

\[
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\[
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\[
\text{for the equations of motion of the spacecraft regarding the nontrivial solution, we can write}
\]

\[
\text{as well as a set of equations similar to Eq. (4) for } \dot{v}_i(x_i(t))\]
is quadratic in the generalized velocities
\[ T_{ii} = \sum_{j=1}^{3} \sum_{k=1}^{3} \frac{\partial^2 L}{\partial \theta_j \partial \theta_k} \theta_j \theta_k + \sum_{j=1}^{3} \left[ \sum_{l=1}^{3} \frac{\partial^2 L}{\partial \theta_j \partial \theta_l} \theta_j \dot{\theta}_l \right] \dot{\theta}_j + \sum_{j=1}^{3} \left[ \sum_{l=1}^{3} \frac{\partial^2 L}{\partial \theta_j \partial \theta_l} \theta_j \ddot{\theta}_l \right] + \frac{\partial^2 L}{\partial \theta_j \partial \theta_k} \theta_j \dot{\theta}_k \right] \dot{\theta}_j + \frac{\partial^2 L}{\partial \theta_j \partial \theta_k} \theta_j \ddot{\theta}_k \right] \left( \dot{\theta}_j \right) 
\] is linear in the generalized velocities, and
\[ T_{ij} = \sum_{j=1}^{3} \sum_{k=1}^{3} \frac{\partial^2 L}{\partial \theta_j \partial \theta_k} \theta_j \dot{\theta}_k + \sum_{j=1}^{3} \left( \sum_{l=1}^{3} \frac{\partial^2 L}{\partial \theta_j \partial \theta_l} \theta_j \dot{\theta}_l \right) \dot{\theta}_j + \sum_{j=1}^{3} \left( \sum_{l=1}^{3} \frac{\partial^2 L}{\partial \theta_j \partial \theta_l} \theta_j \ddot{\theta}_l \right) + \frac{\partial^2 L}{\partial \theta_j \partial \theta_k} \theta_j \dot{\theta}_k \right] \dot{\theta}_j + \frac{\partial^2 L}{\partial \theta_j \partial \theta_k} \theta_j \ddot{\theta}_k \right] \left( \dot{\theta}_j \right) 
\]

is free of generalized velocities.

In view of the above, the perturbed Lagrangian can be written in the general functional form
\[ L = L(\theta_j, \dot{\theta}_j, v_i, w_i, \ldots, \theta_{ij}, \dot{\theta}_{ij}, v_{ij}, w_{ij}, \ldots) \quad j = 1, 2, 3, \quad i = 1, 2, \ldots, n \]

Consequently, using Hamilton's principle, the variational equations can be written in the form of Lagrange's equations for the rotational motion
\[ \frac{\partial L}{\partial \theta_j} - \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}_j} = 0 \quad j = 1, 2, 3 \]

as well as the equations for the elastic motion
\[ \frac{\partial^2 L}{\partial \theta_j \partial \theta_k} - \frac{d}{dt} \frac{\partial^2 L}{\partial \dot{\theta}_j \partial \dot{\theta}_k} = 0 \quad j, k = 1, 2, 3 \]

where \( v_i \) is subject to the boundary conditions
\[ \frac{\partial L}{\partial \theta_j} - \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}_j} \bigg|_{\theta_j = \theta_j^0} = 0 \quad \theta_j = 0 \]

is free of generalized velocities.

The variational equations discussed in the preceding section constitute a set of hybrid differential equations, in the sense that the equations for the rotational motion are ordinary differential equations and those for the elastic displacements are partial differential equations, where the latter are subject to given boundary conditions. It will prove convenient to transform the system into one consisting of ordinary differential equations alone. This can be done by using a discretization procedure based on the assumed-modes method (see Ref. 7, Sec. 6-5). Indeed, let us introduce the notation
\[ \theta_j(t) = q_j(t), \quad j = 1, 2, 3 \]

\[ v_{ij}(x, t) = \sum_{j=1}^{3} \phi_j(x) q_j(t) \]
where \( \psi_j(x_j) \) and \( \phi_j(x_j) \) are admissible functions, taken as the eigenfunctions of the fixed-base member. With this notation, Eq. (7) can be written in the matrix form

\[
T_{ij} = \frac{1}{2} \{ q(t) \}^T [m] \{ q(t) \}
\]

where \([ m ]\) is a constant symmetric matrix. On the other hand, Eq. (8) leads to the matrix form

\[
T_{il} = \{ q(t) \}^T [f] \{ q(t) \}
\]

where \([ f ]\) is a constant square matrix. Moreover, from Eq. (9), we can write

\[
T_{0i} = -V_i = -\frac{1}{2} \{ q(t) \}^T [k] \{ q(t) \}
\]

so that, introducing the notation

\[
[g] = [f]^T - [f]
\]

where \([ g ]\) is a skew symmetric matrix, \([ g ]^T = -[ g ]\), we obtain

\[
[m] \{ q \} + [g] \{ q \} + [k] \{ q \} = 0
\]

so that, introducing the notation

\[
[g] = [f]^T - [f]
\]

where \((x(t)}\) and \((y(t)}\) are admissible functions, taken as the eigenfunctions of the fixed-base member. With this notation, Eq. (7) can be written in the matrix form

\[
T_{ij} = \frac{1}{2} \{ q(t) \}^T [m] \{ q(t) \}
\]

where \([ m ]\) is a constant symmetric matrix. From Eqs. (20), we conclude that the eigenvalue problem defined by \([ I ]\) and \([ G ]\) yields both the real part \([ y ]\), and the imaginary part \([ z ]\), of the eigen-

\[
\{ x(t) \} = \{ q(t) \} + i\{ q(t) \}
\]

No confusion should arise from denoting the state vector by \([ x(t)}\), because the symbol \( x \) used to denote the position of a point in the elastic members represents a spatial coordinate independent of time and not a time-dependent generalized coordinate. Accordingly, if we introduce the \( 2N \times 2N \) matrices

\[
[I] = \begin{bmatrix} [m] & [0] \\ [0] & [k] \end{bmatrix}, \quad [G] = \begin{bmatrix} [g] & [k] \\ [-k] & [0] \end{bmatrix}
\]

then the set of \( N \) equations (21) can be transformed into a set of \( 2N \) first-order equations having the matrix form

\[
[I] \{ \dot{x}(t) \} + [G] \{ x(t) \} = 0
\]

where \([ I ]\) is symmetric and \([ G ]\) is skew symmetric, \([ I ] = [I]^T\), \([ G ] = -[G]^T\), because \([ m ]\) and \([ k ]\) are symmetric and \([ g ]\) is skew symmetric.

The matrix equation (24) is of the special form treated in Ref. 1, so that the eigenvalue problem can be solved by the method developed there. Hence, letting \( \{ x(t) \} = e^{\lambda t} \{ x \} \), where \( \lambda \) and \( \{ x \} \) are constant, we obtain the eigenvalue problem

\[
\lambda [I] \{ x \} + [G] \{ x \} = 0
\]

It is shown in Ref. 1 that the solution of the eigenvalue problem (25) consists of \( 2N \) eigenvalues \( \lambda \) and eigenvectors \( \{ x \} \), \((r = 1, 2, ..., 2N)\), where the eigenvalues consist of pairs of pure imaginary complex conjugates, \( \lambda = \pm i \omega \), and the eigenvectors also consist of pairs of associated complex conjugates \( \{ x \} \), and \( \{ z \} \), \((r = 1, 2, ..., N)\). Moreover, the eigenvectors are orthogonal in a certain sense. Reference 1 develops a method whereby the eigenvalue problem can be solved in terms of real quantities alone. The method is summarized in Sec. VII.

**VII. A Method for the Solution of the Eigenvalue Problem**

In solving the eigenvalue problem, Eq. (25), it is desirable to work with real quantities instead of complex quantities. Reference 1 develops a method of solution addressing itself to this very problem. We shall not repeat here the derivation details, but summarize instead the main features.

Assuming the solution \( \lambda = i \omega, \{ x \} = \{ y \} + i \{ z \} \), where \( \{ y \} \), and \( \{ z \} \), are the real and imaginary parts of the eigenvector, respectively, and separating the real and imaginary parts of the resulting equation, we obtain two algebraic equations that can be reduced to

\[
\omega^2 [I] \{ y \} = [K] \{ y \}, \quad \omega^2 [I] \{ z \} = [K] \{ z \}, \quad r = 1, 2, ..., N
\]

where

\[
[K] = [G]^T [I]^{-1} [G] = [K]^T
\]

is a real symmetric matrix. From Eqs. (26), we conclude that the eigenvalue problem defined by \([ I ]\) and \([ K ]\) yields both the real part \([ y ]\), and the imaginary part \([ z ]\), of the eigen-
The eigenvalue problem associated with a real symmetric matrix can be solved using various methods. In practice, the existence of very close eigenvalues and multiple eigenvalues affects the number of iterations required in solving the problem.

Useful algorithms include Householder's method, the QR algorithm, and inverse iteration. These are found in many texts on numerical analysis, such as that by Ralston and Rabinowitz. The orthogonality property of the eigenvectors is essential to various computational methods.

The most important one is the orthogonality property, with the implied independence of the eigenvectors. This involves solving the eigenvalue problem (26) on the left by \( \mathbf{I}^{-1/2} \mathbf{y} = [\mathbf{K}'] [\mathbf{y}] \), and \( \mathbf{I}^{-1/2} \mathbf{z} = [\mathbf{K}'] [\mathbf{z}] \), respectively.

Using these transformations, the actual eigenvectors are obtained by inserting \( [\mathbf{y}] \) and \( [\mathbf{z}] \), back into transformations (28).

Moreover, using Eq. (30), we obtain

\[
[\mathbf{K}'] = \begin{bmatrix}
[m]^{-1/2} [g] & [k] & [m]^{-1/2} [k] \\
[k] & [0] & [k]^{-1/2} \\
[0] & [k]^{-1/2} & [0]
\end{bmatrix}
\times
\begin{bmatrix}
[m]^{-1/2} [0] \\
[0] \\
[k]^{-1/2}
\end{bmatrix}
\]

where

\[
\mathbf{A} = [m]^{-1/2} [g] [m]^{-1/2}, \quad \mathbf{B} = [k]^{-1/2} [m]^{-1/2}
\]

Note that, to produce the matrix \( [\mathbf{K}'] \), it is only necessary to obtain \( [m]^{-1/2} \) and \( [k]^{-1/2} \). This involves solving the eigenvalue problems associated with \( [m] \) and \( [k] \), respectively.

A large variety of algorithms for the solution of the eigenvalue problem associated with a real symmetric matrix can be found in many texts on numerical analysis, such as those by Ralston and Wilkinson. Useful algorithms include the power method, Jacobi’s method, and inverse iteration. The experience of these authors with Jacobi’s method indicates that, in practice, the existence of very close and multiple eigenvalues affects the number of iterations required to reduce the off-diagonal elements below a prescribed level to a lesser extent than one might expect. For example, in the case of matrices of order up to 50 and for word lengths of 32 to 48 binary digits, the total number of sweeps has averaged about 5 to 6, and the number of iterations required on matrices having multiple eigenvalues has often proved to be somewhat lower than the average. Moreover, even when the matrix has some pathologically close eigenvalues, the corresponding eigenvectors are almost exactly orthogonal. It is evident that they span the correct subspace and give full digital information. This is, perhaps, the most satisfactory feature of Jacobi’s method. Moreover, the extreme simplicity of this method from the programming point of view makes it very desirable. The results presented in Sec. VIII were obtained by this method. It should be stressed, however, that the solution of the eigenvalue problem for \( [K'] \) is not restricted in any way to Jacobi’s method; indeed, it can be obtained by any other method, such as one of those listed above, or a combination thereof.

### VIII. Illustrative Example: The RAE/B Satellite

The general theory developed has been used to calculate the natural frequencies and natural modes of oscillation about the nontrivial equilibrium of the RAE/B satellite. The satellite, shown in Fig. 1, consists of a rigid hub with two pairs of cantilevered radial booms (numbered 1, 2, 3, 4) possessing tip masses, and one pair of cantilevered damper booms (numbered 5, 6). The spacecraft is gravity-gradient stabilized. The analysis consists of: a) the determination of the nontrivial equilibrium; b) the calculation of the matrix coefficients \( m_i, g_{ij}, \) and \( k_{ij} \); and c) the solution of the eigenvalue problem. Explicit expressions for the nontrivial equilibrium and the perturbed Lagrangian for the RAE/B satellite have been derived in Ref. 6, so they will be omitted here for brevity. The parameters used are as follows

\[
\rho_1 = 87.74 \text{ slug ft}^{-2}, \quad B_o = 83.74 \text{ slug ft}^{-2}, \quad C_o = 18 \text{ slug ft}^{-2}, \quad \rho_5 = 4.348 \times 10^4 \text{ slug ft}^{-4}
\]

\[
\rho_3 = \rho_5 = 4.596 \times 10^4 \text{ slug ft}^{-4}
\]

\[
m_1 = m_2 = m_3 = m_4 = 2.40 \times 10^3 \text{slug}, \quad m_5 = m_6 = 0
\]

\[
t_1 = t_2 = t_3 = t_5 = 750 \text{ ft}, \quad t_4 = t_6 = 315 \text{ ft}
\]

\[
E_l = E_{l2} = E_{l3} = E_{l4} = 15.278 \text{ lb ft}^2, \quad \alpha = 30^\circ
\]

\[
E_l = E_{l2} = E_{l3} = E_{l4} = 13.889 \text{ lb ft}^2, \quad \beta = 25^\circ
\]

\[
h_{l1} = h_{l2} = 0.973 \text{ ft}, \quad h_{l2} = h_{l3} = 0.878 \text{ ft}, \quad h_{l3} = h_{l4} = 0
\]

\[
h_{l4} = h_{l5} = 0.705 \text{ ft}, \quad h_{l5} = -h_{l6} = -0.760 \text{ ft}
\]

Note that the damper booms in the actual RAE/B satellite are supported by torsional springs instead of being cantilevered. The difference in the system inaccuracy, for Jacobis’ lowest and indeed, it can be obtained by any other method, such as one of those listed above, or a combination thereof.
To evaluate the elastic displacements $v_{\phi}(x_i)$ and $w_{\phi}(x_i)$ ($i = 1, 2, \ldots, 6$) we assume the solution of Eqs. (3) and (4) in the form

$$v_{\phi}(x_i) = \sum_{i=1}^{n} a_i \phi_i(x_i) \quad w_{\phi}(x_i) = \sum_{i=1}^{n} b_i \phi_i(x_i)$$

$$i = 1, 2, \ldots, n \quad (34)$$

where

$$\phi_i(x_i) = A_i \left[ (\cos \beta_i x_i + \cosh \beta_i x_i) (\sin \beta_i x_i - \sinh \beta_i x_i) 
- (\sin \beta_i x_i + \sinh \beta_i x_i) (\cos \beta_i x_i - \cosh \beta_i x_i) \right] \quad (35)$$

are eigenfunctions corresponding to a bar in bending with the end $x_i = 0$ fixed and having a mass $m_i$ attached at the end $x_i = l_i$. The eigenvalues $\beta_i l_i$ are solutions of the characteristic equation

$$(1 + \cos \beta_i x_i \cosh \beta_i x_i) = \beta_i l_i (m_i / \rho_i l_i) (\sin \beta_i l_i \cosh \beta_i l_i 
- \cos \beta_i l_i \sinh \beta_i l_i) \quad (36)$$

Moreover, the amplitudes $A_i$, are such that the eigenfunctions $\phi_i(x_i)$ are orthonormal, i.e., they satisfy the relation

$$\int_0^l \rho_i \phi_i(x_i) \phi_j(x_i) \, dx_i + m_i \phi_i(x_i) \phi_j(x_i) = \delta_{ij} \quad (37)$$

where $\delta_{ij}$ is the Kronecker delta. Limiting the series in Eqs. (34) to two terms, $p = 2$, the first two roots of Eq. (36) and the amplitudes $A_i$ corresponding to $i = 1, 2, 3, 4$ are

$$\beta_{1} l_{1} = 1.86154 \quad A_{1} = 0.425880 \text{ slug}^{-1/2}$$
$$\beta_{2} l_{2} = 4.66095 \quad A_{2} = 0.033648 \text{ slug}^{-1/2}$$

The first two roots of Eq. (36) with $m_i = 0$, and the amplitudes $A_i$ and $A_j$ corresponding to $i = 5, 6$ are

$$\beta_{3} l_{3} = 1.87511 \quad A_{3} = 0.63510 \text{ slug}^{-1/2}$$
$$\beta_{4} l_{4} = 4.69414 \quad A_{4} = 0.04899 \text{ slug}^{-1/2}$$

By using the technique developed in Ref. 6, we obtain the final results given in Table 1.

It will prove of interest to list the elastic displacements of the end points. These displacements are

$$v_{10}(l_{1}) = -140.95 \text{ ft} \quad w_{10}(l_{1}) = 1.8990 \text{ ft}$$
$$v_{20}(l_{2}) = 140.92 \text{ ft} \quad w_{20}(l_{2}) = 1.8993 \text{ ft}$$
$$v_{30}(l_{3}) = -140.92 \text{ ft} \quad w_{30}(l_{3}) = -1.8992 \text{ ft}$$
$$v_{40}(l_{4}) = 140.95 \text{ ft} \quad w_{40}(l_{4}) = -1.8990 \text{ ft}$$
$$v_{50}(l_{5}) = -0.048677 \text{ ft} \quad w_{50}(l_{5}) = 0.86067 \text{ ft}$$
$$v_{60}(l_{6}) = -0.048669 \text{ ft} \quad w_{60}(l_{6}) = 0.86067 \text{ ft}$$

B. Calculation of the Matrix Coefficients

Using Eqs. (A1)-(A19) and Eq. (20), in conjunction with the above data, we obtain the elements of the matrices $[m]$, $[g]$, and $[k]$. Lack of space prevents listing the matrix elements. Then, using Eq. (32), we obtain the matrix $[K']$. The elastic displacements, measured from equilibrium, were represented by only one admissible function each, resulting in two degrees of freedom per boom. Experience has shown that the addition of another admissible function yields only a small change in the system natural frequencies.

C. Solution of the Eigenvalue Problem

Solving the eigenvalue problem associated with $[K']$ by Jacobi's method, we obtain the natural frequencies squared given in Table 2.
Table 1 Constants defining nontrivial equilibrium

<table>
<thead>
<tr>
<th>i</th>
<th>( a_{ij} )</th>
<th>( a_{2i} )</th>
<th>( b_{ij} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.41093 \times 10^2</td>
<td>-0.30339</td>
<td>0.55781</td>
</tr>
<tr>
<td>2</td>
<td>-0.41086 \times 10^2</td>
<td>0.30224</td>
<td>0.55789</td>
</tr>
<tr>
<td>3</td>
<td>-0.41085 \times 10^2</td>
<td>-0.30324</td>
<td>-0.55788</td>
</tr>
<tr>
<td>4</td>
<td>-0.41093 \times 10^2</td>
<td>0.30339</td>
<td>-0.88536 \times 10^{-2}</td>
</tr>
<tr>
<td>5</td>
<td>-0.93895 \times 10^2</td>
<td>-0.12898 \times 10^3</td>
<td>0.16439</td>
</tr>
<tr>
<td>6</td>
<td>-0.93881 \times 10^2</td>
<td>-0.12894 \times 10^3</td>
<td>0.16439</td>
</tr>
</tbody>
</table>

Table 2 Natural frequencies squared

<table>
<thead>
<tr>
<th>i</th>
<th>( \omega_i^2 \times 10^6 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.29648 \times 10^{-1}</td>
</tr>
<tr>
<td>2</td>
<td>4.57848 \times 10^{-1}</td>
</tr>
<tr>
<td>3</td>
<td>8.02608 \times 10^{-1}</td>
</tr>
<tr>
<td>4</td>
<td>1.44713</td>
</tr>
<tr>
<td>5</td>
<td>1.44713</td>
</tr>
<tr>
<td>6</td>
<td>1.44713</td>
</tr>
<tr>
<td>7</td>
<td>1.44713</td>
</tr>
<tr>
<td>8</td>
<td>1.44713</td>
</tr>
<tr>
<td>11</td>
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<tr>
<td>12</td>
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<td>13</td>
<td>1.46268</td>
</tr>
<tr>
<td>14</td>
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</tr>
<tr>
<td>15</td>
<td>2.00414</td>
</tr>
<tr>
<td>16</td>
<td>2.00414</td>
</tr>
<tr>
<td>17</td>
<td>2.14185</td>
</tr>
<tr>
<td>18</td>
<td>2.14185</td>
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<td>19</td>
<td>2.14185</td>
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<tr>
<td>28</td>
<td>3.77371</td>
</tr>
<tr>
<td>29</td>
<td>1.25320 \times 10^{-2}</td>
</tr>
</tbody>
</table>

The modal vectors have also been obtained, but lack of space does not permit their listing. As a matter of interest, however, the first two modes are plotted in Figs. 2 and 3. Because these two modes correspond to the same eigenvalue \( \omega_i^2 \), any linear combination of the two modes is also a mode (see Ref. 1). This fact can be used at times to simplify the vectors.

IX. Conclusions

This paper presents a method for the determination of the natural frequencies and mode shapes of rotating structures oscillating about nontrivial equilibrium. The formulation of the eigenvalue problem involves the derivation of the variational equations about the nontrivial equilibrium and the discretization of these equations. The linearized discrete system is defined by two symmetric matrices and one skew symmetric matrix. Hence, it belongs to the class of linear gyroscopic systems. Using a technique developed by Meirovitch\(^1\) the eigenvalue problem is transformed into one defined by a single real symmetric matrix, the solution of which can be obtained by a large variety of algorithms. The approach is suitable for the solution of high-order systems.

As an illustration, the problem associated with a 15-degree-of-freedom system is solved. Both natural frequencies and modal vectors for the complete structure are obtained. The mathematical model simulates the RAE/B satellite, a gravity-gradient stabilized flexible spacecraft.

Appendix

The elements of the matrices \([m]\), \([f]\), and \([k]\) are as follows

\[
m_{jk} = \frac{\partial^2 L_i}{\partial \theta_j \partial \theta_k} + \frac{\partial^2 L_i}{\partial \dot{\theta}_j \partial \dot{\theta}_k} \quad j,k=1,2,3
\]  

\[
f_{jk} = \frac{\partial^2 L_i}{\partial \dot{\theta}_j \partial \theta_k} \quad j,k=1,2,3
\]  

\[
f_{jk} = \frac{\partial^2 L_i}{\partial \theta_j \partial \dot{\theta}_k} \quad j,k=1,2,3
\]
\[
i = 1, 2, \ldots, n
\]
\[
f_{jk} = \int_{0}^{\pi} \frac{\partial^2 L_{ij}}{\partial \theta_{ij} \partial \theta_{kj}} \phi_j(x_i) \, dx_i + \left[ \frac{\partial^2 L_{ij}}{\partial \theta_{ij} \partial v_{00}} \phi_j(x_i) \right] x_i = q
\]
\[
j = 2(i-1)p + 4, 2(i-1)p + 5, \ldots, (2i-1)p + 3
\]
\[
i = 1, 2, \ldots, n; k = 1, 2, 3
\]
\[
f_{jk} = \int_{0}^{\pi} \frac{\partial^2 L_{ij}}{\partial v_{00} \partial w_{00}} \phi_j(x_i) \, dx_i + \left[ \frac{\partial^2 L_{ij}}{\partial v_{00} \partial w_{00}} \psi_j(x_i) \right] x_i = q
\]
\[
j = (2i-1)p + 4, (2i-1)p + 5, \ldots, 2ip + 3
\]
\[
i = 1, 2, \ldots, n; k = 1, 2, 3
\]
\[
k_{jk} = \int_{0}^{\pi} \frac{\partial^2 L_{ij}}{\partial w_{00} \partial \theta_{ij}} \phi_k(x_i) \, dx_i + \left[ \frac{\partial^2 L_{ij}}{\partial w_{00} \partial \theta_{ij}} \phi_k(x_i) \right] x_i = q
\]
\[
j = 2(i-1)p + 4, 2(i-1)p + 5, \ldots, (2i-1)p + 3
\]
\[
k_{jk} = \int_{0}^{\pi} \frac{\partial^2 L_{ij}}{\partial \theta_{ij} \partial v_{00}} \psi_k(x_i) \, dx_i + \left[ \frac{\partial^2 L_{ij}}{\partial \theta_{ij} \partial v_{00}} \psi_k(x_i) \right] x_i = q
\]
\[
j = 2(i-1)p + 4, (2i-1)p + 5, \ldots, 2ip + 3
\]
\[
i = 1, 2, \ldots, n
\]

References