Closed-Form Solutions for a Class of Optimal Quadratic Regulator Problems With Terminal Constraints

Closed-form solutions are derived for coupled Riccati-like matrix differential equations describing the solution of a class of optimal finite time quadratic regulator problems with terminal constraints. Analytical solutions are obtained for the feedback gains and the closed-loop response trajectory. A computational procedure is presented which introduces new variables for efficient computation of the terminal control law. Two examples are given to illustrate the validity and usefulness of the theory.

Introduction

Many engineering applications involve maneuvering a system between two quiescent or known moving states. These applications have motivated the research and development of a number of optimal maneuver strategies. Traditionally, the approach in treating the maneuver problem is typified by optimal control theory which leads to a system of coupled state-costate differential equations [1-6], describing a two-point boundary value problem. The boundary conditions are usually divided with half specified at the initial time, and the rest at the final time. Several common tools for attacking this type of problem exist including the use of direct numerical integration, the automatic synthesis program (ASP) matrix iteration technique [7] and the negative exponential method [8]. The main cost of the direct numerical integration is the computational burden, particularly when the number of differential equations to be integrated is large. The ASP matrix iteration procedure which uses the transition matrix of the state-costate equations is usually satisfactory. However, for a higher order system with high frequency modes, the ASP procedure may require so small a time increment that total number of iterations is very costly in computation time. In such cases, the simulation time may be substantially longer than the real time for control maneuvers, making real time control impossible. The negative exponential method is to convert the spectral form of the transition matrix of the coupled state-costate equations from the initial value form into a two-point boundary value form. Such conversion eliminates the numerical difficulties caused by unstable exponentials in the transition matrix. The same idea has been used in reference [9] to develop a nonrecursive algebraic solution for the discrete Riccati equation. The solution is analogous to that in reference [8] for the continuous Riccati equation. Both solutions involve finding the eigenvalues and eigenvectors of the canonical state-costate equations. The most difficult part of the negative exponential method is the computation of eigenvalues and eigenvectors of a matrix with the order twice higher than the order of the system.

In contrast to the classical methods, one other method exists [10-12], which transforms the standard Riccati equation into a form that eliminates the nonlinear quadratic term by expressing the solution in terms of the steady-state and time varying transient terms. The transient term is solved in closed form without using the transition matrix of the state-costate equations. The major drawback, as is common with penalized terminal states used in these papers, is numerical difficulties associated with handling a very large terminal weighting matrix. Furthermore, repeated simulation of a complex system over various terminal weighting matrix ranges can be very costly and sometimes not feasible.

This paper describes three new developments related to a class of optimal finite time quadratic regulator problems with terminal constraints instead of penalizing the terminal states. The transition matrix of the coupled state-costate equations used in references [7-9] is not used in these developments. First, analytical solutions are presented for time varying gains given by three coupled Riccati-like matrix differential equations for a terminal controller. Second, a closed form expression for the closed-loop system state trajectory is developed. Third, a set of new formulations is described for the computation of control laws. The new formulation leads to an alternate representation of the terminal control problem. Computation of the time varying gains is based on matrix transformations which reconstruct the three coupled nonlinear matrix differential equations into two uncoupled linear matrix differential equations and one algebraic matrix equation. It is not necessary to specify all the terminal states.

Examples including a first-order system and a simple spacecraft are used to demonstrate the validity of all the analytical solutions developed in this paper. In the second example representing an one-axis spinning spacecraft, the system state is augmented to include the control, which enables the
penalization of the control rate. Rest-to-rest and spin-up maneuvers for the spacecraft will be shown.

**Optimal Terminal Controller**

The optimal terminal control problem is formulated by finding the control inputs $u(t)$ to minimize the typical cost function

$$J = \frac{1}{2} \int_{t_0}^{t_f} (x^T F x + u^T R u) dt$$

for the system

$$\dot{x} = Ax + Bu, \text{ given } x(t_0)$$

with outputs

$$y = Fx$$

which is subject to the specified terminal constraints at time $t_f$

$$x_i(t_f) = \hat{x}_i, \quad i = 1, \ldots, q; \quad q \leq n$$

where $x$ is the state vector, $u$ is the control vector, $A$ is the system dynamics matrix, $B$ is the control influence matrix, $F$ is the measurement influence matrix, $Q = Q^T \geq 0$ is the output weighting matrix and $R = R^T > 0$ is the control weighting matrix.\(^{1}\) It is assumed that the pair $(A^T, F^T)$ is completely observable and $[A, B]$ stabilizable. Of particular interest is the fact that the performance index of equation (1) does not contain a terminal weight matrix, which penalizes the final values of the state. Indeed, it is the principal intent of this paper to present a numerical technique which enforces the satisfaction of the constraints of equation (4), without incurring the numerical difficulties associated with a very large terminal weighting matrix approach for the problem [10-12].

As shown in reference [1], the necessary conditions defining the optimal solution are given by the following coupled Riccati-like matrix differential equations

$$\dot{P} + PA + A^T P - PB^{-1} B^T P + F^T Q F = 0; \quad P(t_0) = 0$$  \hspace{1cm} (5)

$$\dot{S} + (A^T - PB^{-1} B^T)S = 0; \quad S(t_0) = 0$$  \hspace{1cm} (6)

$$\dot{G} = S^T B R^{-1} B^T G; \quad G(t_0) = 0$$  \hspace{1cm} (7)

where $\psi = [\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_q]$ is the terminal constraints on the state vector and the optimal continuous feedback law is given by

$$u(t) = -C(t)x(t) - D(t)\psi$$  \hspace{1cm} (8)

and

$$C = R^{-1} B^T (P - S G^{-1} S^T); \quad D = R^{-1} B^T S G^{-1}$$  \hspace{1cm} (9)

(10)

The control vector $u$ will take the state vector from $x(t_0)$ at time $t_0$ to $\psi$ at time $t_f$ while minimizing the cost function of equation (1).

In order to compute $C(t)$ and $D(t)$ efficiently in equations (9) and (10), closed-form solutions for equations (5)-(7) are presented, thus reducing the solution for the control problem to the direct computation of algebraic equations without numerical integration. The Riccati equation (5) can be solved by several different methods such as shown in references [7-12]. Here, the method shown in references [10-12] is used to obtain the solution of equation (5). Let

$$P(t) = P_{ss} + Z^{-1}(t)$$  \hspace{1cm} (11)

where $P_{ss}$ is the positive definite solution for the algebraic Riccati equation

$$-A^T P_{ss} - P_{ss} A + P_{ss} B R^{-1} B^T P_{ss} - F^T Q F = 0$$  \hspace{1cm} (12)

For a completely observable and stabilizable system [2], $P_{ss}$ and $Z^{-1}(t)$ exist for $t_f \geq t \geq t_0$. The differential equation for $Z(t)$ is given by

$$\dot{Z} = \bar{A} Z + Z \bar{A}^T - B R^{-1} B^T; \quad Z(t_f) = \bar{P}_{ss}$$  \hspace{1cm} (13)

with the solution

$$Z(t) = Z_{ss} e^{\bar{A}(t-t_0)} (Z_{ss} + P_{ss}) e^{-\bar{A} (t_0-t_0)}$$  \hspace{1cm} (14)

where $e^{(\cdot)}$ is the exponential matrix, $\bar{A} = A - BR^{-1} BT$ is a stability matrix with eigenvalues having negative real parts and $Z_{ss}$ satisfies the algebraic Lyapunov equation

$$\bar{A} Z_{ss} + Z_{ss} \bar{A}^T = B R^{-1} B^T$$  \hspace{1cm} (15)

Obviously, there exists a continuous solution for the square matrix $Z(t)$.

The new solution for the rectangular time-varying matrix $S(t)$ in equation (6) follows on assuming the product form solution

$$S(t) = Z^{-1}(t) S_{ss}(t)$$  \hspace{1cm} (16)

Substitution of equation (16) into equation (6) with the aid of equation (13) leads to the following linear matrix differential equation for $S_{ss}(t)$:

$$\dot{S}_{ss}(t) - \bar{A} S_{ss}(t) = 0; \quad S_{ss}(t_f) = -P_{ss} S(t_f)$$  \hspace{1cm} (17)

The solution for $S_{ss}(t)$ is

$$S_{ss}(t) = e^{\bar{A}(t-t_0)} P_{ss} S(t_f)$$  \hspace{1cm} (18)

Now from equation (7), making use of equations (6) and (13), the following solution for $G(t)$ is obtained:

$$G(t) = S^T(t) Z(t) S(t) + S^T(t_f) P_{ss} S(t_f)$$  \hspace{1cm} (19)

which can be easily verified by direct differentiation of equation (19). Note that [4] $G(t) < 0$ for $t_f \geq t \geq t_0$ since $G(t) \geq 0$ and $G(t_f) = 0$.

**Closed-Form Solution for the State Trajectory**

Introducing the control input $u$ into equation (2) yields the closed-loop system dynamics equation as follows

$$\dot{x} = [\bar{A} - BR^{-1} B^T Z_{ss} \psi] x - BR^{-1} B^T S_{ss} \psi$$  \hspace{1cm} (20)

where, with the aid of the matrix inversion lemma [1],

$$Z_{ss}(t) = [Z^{-1}(t) - S(t) G^{-1}(t) S^T(t)]^{-1}$$  \hspace{1cm} (21a)

and

$$S_{ss}(t) = S(t) G^{-1}(t)$$  \hspace{1cm} (21b)

It is obvious that $Z_{ss}(t)$ exists for $t \geq t \geq t_0$ since $Z(t), S(t)$ and $W$ exist. Direct differentiation of equation (21) leads to

$$\dot{Z}_{ss} = -Z_{ss} \bar{A}^T - (\bar{A} Z_{ss}) + B R^{-1} B^T$$  \hspace{1cm} (22)

To obtain the homogeneous solution $x_h$ for equation (20), consider the following coordinate transformation for the variable $x_h$

$$x_h(t) = Z_{ss}(t) r(t)$$  \hspace{1cm} (23)

where $r(t)$ is a vector function to be determined. Differentiating equation (23) in combination with equation (22), and inserting into equation (20) with $\psi = 0$, leads to

$$\dot{Z}_{ss}(t) = 0$$  \hspace{1cm} (24)

Choose $r(t)$ to satisfy

$$\dot{r} + \bar{A}^T r = 0$$  \hspace{1cm} (25)

which possesses the solution

$$r = e^{-\bar{A}^T (t-t_0)} r_0$$  \hspace{1cm} (26)

where $r_0$ is an integration constant vector. Let the dependent state vector $x$ be

$$x(t) = \Phi(t) (r_0 + x_h(t))$$  \hspace{1cm} (27)

\(^{1}\)For maneuvers where the state is augmented by the control and control-rate penalties, the matrices $A$ and $B$ are modified as shown in reference [11].
where

\[ \Phi(t) = Z_m(t)e^{-A^T(t-t_m)} \]  

(28)

In view of equations (20) and (22), it is easy to show that

\[ \Phi(t) = [\bar{A} - BR - B'^T Z_m^{-1}(t)]\Phi(t) \]  

(29)

Let \( \Phi_i \) be the inverse of \( \Phi \). Note that \( \Phi \) is a nonsingular matrix for \( t_f \geq t \geq t_0 \). Pre and post multiplying equation (29) by \( \Phi_i \) lead to

\[ \Phi_i(t) = -\Phi_i(t)[\bar{A} - BR - B'^T Z_m^{-1}(t)] \]  

(30)

Substituting equation (27) into equation (20) with the aid of equation (29) yields

\[ x_p(t) = -\Phi_i(t)BR - B'^T S_m(t) \psi \]  

(31)

Now making use of equation (30) and the following formulation for \( S_m \):

\[ dS_m/dt = [-\bar{A}^T + Z_m^1BR - B'^T]S_m \]  

(32)

Equation (31) can be solved by

\[ x_p(t) = [\Phi_i(t)Z_m(t)S_m(t) + X_{\Phi}] \psi \]  

(33)

where \( X_{\Phi} \) is an integration constant matrix to be determined by the initial condition. Equations (32) and (33) can be easily verified by direct differentiation. Combination of equations (29) and (32) yields

\[ d[\Phi^T(t)S_m(t)]/dt = 0 \]  

i.e.

\[ \Phi^T(t)S_m(t) = \text{constant}; \]  

(34)

Direct substitution of equation (33) into equation (27) and evaluation of the equation obtained at \( t = t_0 \) provides

\[ x_p(t_0) = 0 \Rightarrow x_{\Phi} = S_m(t_0) \]  

(35)

and

\[ x_p(t_0) = x(t_0) = p_0 = \Phi_i(t_0)x(t_0) \]  

(36)

by noting that \( \psi \) is an arbitrary constant vector. The solution for the state vector \( x \) becomes

\[ x(t) = \Phi(t) \Phi_i(t_0)x(t_0) + [\Phi(t)S_m(t_0) - Z_m(t)S_m(t)]\psi \]  

(37)

For the special case where all the terminal states are specified (i.e. \( q = n \)), \( S \) is a nonsingular square matrix and algebraic manipulation leads to

\[ S_m = -[Z + S_m^1P_m^1S_m^2]^{-1}e^{\bar{A}(t-t_f)} \]  

and

\[ Z_m^{-1} = Z^{-1} - SG^{-1}ST = [Z + S_m^1P_m^1S_m^2]^{-1} \]  

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It follows that

\[ Z_m(t)S_m(t) = -e^{\bar{A}(t-t_f)} \]  

(40)

For single degree of freedom \( S_m \) reduces to a hyperbolic function multiplied by a constant. Equation (37) becomes

\[ x(t) = \Phi(t) \Phi_i(t_0)x(t_0) + [e^{\bar{A}(t-t_f)} + \Phi(t)S_m(t_0)]\psi \]  

(41)

It is immediately seen that, when \( t = t_f \), \( Z_m(t_f) = 0 \) and \( \Phi(t_f) = 0 \) from equations (39) and (28) and thus \( x(t_f) = \psi \).

Practical Computation for the Solutions

The above results allow solutions for a terminal controller with any given dimension (where the dimension is meant the number of the states \( n \) and terminal constraints \( g \)). First, we obtain the solutions (14), (16), (18), and (19) at time \( t \). Then equations (11), (16), and (19) will determine gains \( C \) and \( D \) in equations (9) and (10). The control \( u(t) \) is thus calculated from equation (8) at any time \( t \).

An alternative way to implement these solutions is introduced as follows. Examination of equation (22) reveals that \( Z_m \) can be solved by

\[ Z_m(t) = Z_{ss} + e^{\bar{A}(t-t_f)}[Z_m(t_f) - Z_{ss}]e^{-\bar{A}(t-t_f)} \]  

(42a)

and

\[ Z_m^{-1} = e^{\bar{A}(t-t_f)}[Z_m^{-1}(t_f)Z_{ss}^{-1}e^{\bar{A}(t-t_f)} + Z_m(t_f) - Z_{ss}]^{-1}e^{-\bar{A}(t-t_f)} \]  

(42b)

where \( Z_{ss} \) satisfies the algebraic equation (15) and \( Z_m(t_f) \) is the terminal condition given by

\[ Z_m(t_f) = -P_{ss}^{-1}S(t_f)\{S(t_f)P_{ss}^{-1}S(t_f)\}^{-1}S(t_f)P_{ss}^{-1} \]  

(43)

which can be easily obtained from equation (21). For the case where all terminal states are specified (i.e. \( q = n \) and \( Z_m(t_f) = 0 \) equation (42b) obviously becomes equation (39). Equation (34) suggests that \( S_m \) have the solution

\[ S_m(t) = S(t)G^{-1}(t) = Z_{ss}^{-1}(t)e^{\bar{A}(t-t_f)}Z_m(t_f)S(t_f) \]  

(44)

Now combination of equations (21a) and (21b) with some matrix manipulations leads to

\[ Z_m(t)S_m(t) = S(t)W \]  

(45)

which provides, with the aid of equation (18), the terminal condition

\[ Z_m(t_f)S_m(t_f) = -P_{ss}^{-1}S(t_f)W \]  

(46)

For \( q = n \), equation (45) is replaced by equation (40). Equations (42) and (44) thus constitute the feedback control \( u \) as follows.

\[ u(t) = -R^{-1}B^T[P_{ss} + Z_m^{-1}(t)]x(t) - R^{-1}B^T S(t) \psi \]  

(47)

The solution for the state trajectory is given by equation (37). It is seen that this alternative procedure for the numerical implementation of the terminal controller eliminates the computation \( G^{-1}(t) \). This considerably reduces the computational time and errors particularly when the size of matrix \( G(t) \) is large. Since \( \bar{A} \) is a stability matrix with eigenvalues having negative real parts, the exponential \( e^{\bar{A}(t-t_f)} \) becomes a decay function. The inverse formula for the matrix \( Z_m(t) \) as shown in equation (42b) is recommended to eliminate the numerical difficulties in computing the positive exponential \( e^{\bar{A}(t-t_f)} \) in equation (42a), particularly when the stability matrix \( \bar{A} \) contains high frequency modes. Note that the form of solution for the feedback control \( u \) as shown in equation (47) is different from that developed in references [8 and 9].

An efficient method for the computation of \( e^{\bar{A}(t-t_f)} \) is needed to reduce the computational burden on costs and numerical errors. For numerical simulations, the exponential matrix \( e^{\bar{A}(t-t_f)} \) is propagated forward in time using the relation

\[ e^{\bar{A}[(t+t_m)-t]} = e^{\bar{A}(t-t_m)}e^{-\bar{A}t_m} \]  

(48)

where \( e^{\bar{A}(t-t_f)} \) and \( e^{-\bar{A}t_m} \) are computed only once, using a Padé series expansion approximation which is highly efficient and accurate [13]. When time proceeds, the matrix exponential at any other time will simply be the multiplication of the result at the previous time step.

The inverse of the time-dependent symmetric matrix in equation (42b) becomes the most time consuming operation for the terminal controller. As far as computational speed is concerned, factorization methods developed in references [14–15] are appropriate for this application.

Illustrative Examples

First-Order System. As the first example, given the first-order system

\[ \dot{x} = -x/a + u \]  

(49)

determine a terminal controller that will drive the state \( x \) from its initial position \( x_0 \) to the terminal position \( \psi \).
If \( x^2 \) is aimed to be kept below \( x^2_m \) constant, using \( u^2 \) below \( u^2_m \) constant, the following performance index is considered

\[
2J = \int_0^f (Qx^2 + Ru^2)dt \quad \text{where} \quad Q = x^2_m \quad \text{and} \quad R = u^2_m
\]

which is subjected to the terminal constraints

\[
x(f) = \psi
\]

The corresponding solution \( P_{xf} \) for the steady state Riccati equation (12) is

\[
P_{xf} = R[-1/a + (Q/R + 1/a^2)^{1/2}]
\]

Other related constants are given below

\[
\dot{\psi} = - (Q/R + 1/a^2)^{1/2}
\]

and

\[
Z = -[2R(Q/R + 1/a^2)^{1/2}]^{-1}
\]

Then equations (14) and (18) lead to

\[
Z = Z_m + S \int\int S_d^T = -2Z_m \sinh \dot{\psi}(t - f) e^{\dot{\psi}(t - f)}
\]

which gives

\[
\dot{\psi} = 2Z_m \sinh \dot{\psi}(t - f) e^{\dot{\psi}(t - f)}
\]

Application of equations (38) and (39) to equation (47) thus yields the control input \( u \) for equation (2) as follows

\[
u = -R^{-1} [P_m - e^{-A(t - f)} / 2Z_m \sinh \dot{\psi}(t - f)] x
\]

\[
+ [1/2Z_m \sinh \dot{\psi}(t - f)] \psi
\]

The value of \( x \) at any time \( t \) from equation (41) becomes

\[
x(t) = x(t_0) \sinh \dot{\psi} (t - t_f)/\sinh A(t_0 - t_f)
\]

\[
+ \psi \cosh \dot{\psi} (t - t_f) - \sinh \dot{\psi} (t - t_f) \coth \dot{\psi} (t_0 - t_f)
\]

For \( t = t_f \), the feedback gains \( C \) and \( D \) approach \( \infty \) and the terminal state becomes \( \psi \). Equation (58) can also be obtained by direct integration of equation (20).

Suppose that \( a = 10, Q = R = 1 \). If \( \psi = 0, x(t_0) = 1 \) and \( t_f = 5 \), the control \( u \) and the state trajectory \( x \) are given in Fig. 1. The final state \( x(t_f) \) is zero.

For the second case, if \( x(t_0) = 0 \) and \( \psi = 1 \), Fig. 2 shows the control \( u \) and the state trajectory \( x \) for this case. The final state \( x(t_f) \) is 1.

A Simple Spacecraft. The second example illustrates the
application of solutions derived in this paper to the maneuvering of a simple spacecraft as shown in Fig. 3. The spacecraft comprises two panels clamped symmetrically to a rigid central body. The maneuver to be performed is to keep the roll attitude $\phi$ to a specified angle for fine target pointing, or to spin up the spacecraft to a specified roll angular velocity $\omega$ for orbit stabilization. This maneuver is to be completed in the time interval zero to $t_f$. The control to be used for the maneuver is the torque $u$ which is provided by a control moment gyro attached to the central rigid body. For the purpose of simplicity, assumed that the two panels are rigid. The equations of motion are given below

$$\dot{u} = u_t$$

$$\phi = \omega$$

where $u_t$ is the roll-time constant, $I$ is the total moment of inertia for the spacecraft, $u$ is the control torque provided by the gyro, and $u_t$ is the torque rate. Using quadratic synthesis, consider the minimization of the following performance index

$$2J = \int_0^{t_f} \left( \frac{1}{2} \dot{\phi} \phi \phi + \frac{1}{2} (u - u_{in})^2 + \frac{1}{2} (u_t - u_{in})^2 \right) dt$$

subject to the terminal constraints

$$\psi = x(t_f) = (u, \omega, \phi)$$

where $\phi_{in}$, $u_{in}$, and $u_{in}$ are constants representing weighting coefficients, respectively, for the roll angle $\phi$, the control torque $u$ and the torque rate $u_t$. Including the penalty on the control torque rate $u_t$ allows the control torque $u$ to be specified at the initial and final time; thereby eliminating the initial and terminal jump discontinuities in the control profile. This reduces the excitation of vibrational motion when some parts of the spacecraft are elastically flexible [11].

The analytical solution $P_{ss}$ of the Riccati equation (12) for this case can be found in reference [1], p. 170. The solution $Z_{ss}$ of the algebraic Lyapunov equation (15) can easily be obtained by straightforward manipulation in terms of elements in the matrix $P_{ss}$. Note that $P_{ss}$ and $Z_{ss}$ can be numerically computed [16-17].

Suppose that the moment of inertia for the spacecraft $I = 100 \text{ Kg m}^2$, the roll-time constant $a = 100$ seconds and the weighting constants $\phi_{in} = u_{in} = u_{in} = 1$. For the rest-to-rest case, assume that the initial states $u(0) = \omega(0) = 0$ and $\phi(0) = 1$ radian, and specify that the final states $u(t_f) = \omega(t_f) = \phi(t_f) = 0$, with $t_f = 10$ seconds. This is a rest-to-rest maneuver because the angular velocity $\omega$ and the angular acceleration $\omega$ are zero at the initial time zero and the final time 10 seconds. Figure 4 presents time histories of the control torque $u$, the angular velocity $\omega$, and the roll angle $\phi$ computed from equation (37) in conjunction with equations (42) and (44). The torque rate $u_t$ is also shown in Fig. 4 which shows the jump discontinuities at time zero and 10 seconds. However, the control torque $u$ which is the integral of the torque rate $u_t$ has a smooth curve. The angular velocity $\omega$ represents the spinning rate history for the spacecraft undergoing a 1 radian rotation in 10 seconds subject to some internal resistance with the roll-time constant $a$. No energy remains in the system after the rest-to-rest maneuver.

For the spin-up case, Fig. 5 presents a spin-up maneuver, in which the final angle $\phi$ is free to be determined by the controller, while the initial states are given as $u(0) = \omega(0) = \phi(0) = 0$, and the final states are specified as $u(10) = 0$ and $\omega(10) = 1 \text{ rad/s}$. This case is typical of the problem of increasing the rotation rate of a spacecraft for on-orbit stabilization. After a spin-up maneuver, the spacecraft will gain a kinematic energy due to the nonzero angular velocity $\omega$. This fact can be seen from the time history of the control torque as shown in Fig. 5 by integrating the torque over the time interval. The integration gives a net work after the spin-up maneuver. However, the kinematic energy will be gradually dissipated by the internal friction characterized by the roll-time constant $a$. A spin-up maneuver may be necessary later to maintain a desired spinning rate.

**Concluding Remarks**

The contribution of this paper is the closed-form solutions for the feedback control gains and the state trajectory with terminal constraints on state and time. In addition, a new way of implementing these solutions is developed. This is a particularly interesting result which leads to an alternate representation of the coupled differential matrix Riccati-like equations required to solve feedback gains of the terminal controller. Computational procedures have been considerably simplified to increase the computational speed. Results of example maneuvers are shown demonstrating the validity of the formulations developed in this paper.

**References**


