Analysis of nonlinear dynamical systems using probabilistic representations
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Background

Boeing F/A-18

The Boeing F/A-18 Hornet is the backbone of the United States Navy and Marine Corps aircraft fleets. But the first four models were plagued by unstartable behavior in their flight controllers. In certain cases, when the aircraft made a high angle-of-attack turn, it would enter an out-of-control state nicknamed the “falling-leaf” mode. The mode is characterized by the following behavior (See Fig. 2):

1. Violent oscillations in yaw and pitch axes.
2. Rapid loss of altitude.
3. Difficulty in recovering due to low airspeed.

Several aircraft have been lost when their pilots ejected after triggering this mode.

Despite performing numerous tests and analyses, the designers of the flight controller were entirely unaware this behavior existed. The lack of knowledge about the F/A-18’s behavior was a consequence of the nonlinear nature of the dynamical system used to model the aircraft. A dynamical system is a set of equations of the form:

\[ x = f(x, t) \]

The falling-leaf mode in the F/A-18 reveals several primary problems:

• Nonlinear systems have the potential for chaos.
• Fewer techniques exist to analyze nonlinear systems compared to linear systems.
• Nonlinear systems are often impossible to solve analytically.

Problems

The falling-leaf mode in the F/A-18 reveals several primary problems:

1. Nonlinear dynamical systems are difficult to understand.
2. Linear analysis is only valid for small regions of state space.
3. The flight control law designers were unaware the behavior existed.

A framework has been developed to enable a new level of analysis for nonlinear systems. This framework can address each of these problems.

Probabilistic Framework

Overview

A method has been developed to understand the stability of nonlinear dynamical systems. The basis for this approach is a Markov chain, which is a probabilistic linear representation of the dynamical system. Figure 3 shows the steps of the framework.

Markov Chain

A Markov chain is a probabilistic way to predict future events. It is characterized by a stochastic or Markov matrix, \( M \), that operates under multiplication. In the following formula, \( P \) represents an initial probability distribution over \( n \) distinct states and \( p^* \) represents the distribution after one time step:

\[ p^* =Mp \]

When a sufficient number of iterations have been applied, the system will approach a steady-state condition, where the probability vector remains constant under application of the stochastic matrix:

\[ p^* = Mp^* \]

The vector \( p^* \) is known as the stationary probability vector. It is equivalent to the eigenvector of \( M \) associated with eigenvalue 1. This vector contains a representation of the steady-state nonlinear behavior of the system: the states with high probabilities correspond to the states the nonlinear system approaches in the long term.

Generating the Stochastic Matrix

The stochastic matrix \( M \) is populated according to the following rule: the entry with indices \( i \) and \( j \) is the probability that the system is currently in state \( j \) and reaches state \( i \) in the next time step:

\[ M_{ij} = P(j \rightarrow i) \]

The following is a list of the steps required to generate the stochastic matrix:

1. Discretize domain into a finite rectangular grid at a certain level of resolution.
2. Create geometric representations of each grid cell.
3. Compute volume of cell intersections to determine Markovian probabilities in above equation.
4. Flight control law designers could use this framework to analyze their laws and discover unwanted behavior before the aircraft reaches production.

Results

The Van der Pol oscillator was used to test the probabilistic technique. This nonlinear system contains a stable region known as a limit cycle; in the long term, all solution trajectories converge upon this region and oscillate within it. See Fig. 4a for the phase portrait. The goal was to predict this limit cycle using the probabilistic framework.

Figures 4b–4d demonstrate the computed limit cycle at an increasing resolution. Each cell is shaded according to its stationary probability; cells with higher probabilities have lighter shades. The white outline depicts the true limit cycle of the system.

This prototype demonstrates the validity of using statistical methods to predict the long-term behavior of a nonlinear dynamical system.

Conclusion

The goal of predicting the steady-state behavior of a nonlinear dynamical system has been met. The original problems can now be addressed:

1. With the right tools, nonlinear systems can be transformed into linear ones, making them straightforward to understand.
2. Traditional linear analysis is no longer necessary.
4. Flight control law designers could use this framework to analyze their laws and discover unwanted behavior before the aircraft reaches production.