Analysis of Nonlinear Dynamical Systems Using Markovian Representations

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This paper presents a framework for understanding the behavior of nonlinear dynamical systems in N-dimensional state space. The basis of this approach is the generation of a stochastic matrix using techniques from the field of statistical physics. Though it contains the nonlinear nature of the system, the matrix itself is a linear operator, allowing easy and straightforward computation of the long-term behavior of the system via its stationary probability vector. Such a technique can be applied to situations where knowledge of the stability of highly nonlinear dynamical systems is of critical importance, such as in the design flight control laws.

I. Background

The field of Dynamics and Control is concerned with the study of *dynamical systems.* A dynamical system is "a means of describing how one state develops into another state over the course of time."[1] These systems are usually represented by differential equations for continuous time and difference equations for discrete time. An example dynamical system is the predator-prey model usually studied in an introductory differential equations course. In this system, the independent variables represent the populations of different species.

There are two broad categories of dynamical systems: *linear* systems and *nonlinear* systems. Linear systems are relatively easy to solve and there are many tools and techniques available to analyze them. For example, the predator-prey model has analytical solutions: functions that give the population of each species with respect to time. On the other hand, nonlinear systems are frequently impossible to solve analytically and there are far fewer ways to analyze them. This poses a problem because nonlinear dynamical systems occur far more frequently when dealing with physical phenomena.

Nonlinear dynamical systems are primarily studied by understanding their *stability.* Stability refers to "the robustness of a given outcome to small changes in initial conditions or small random fluctuations."[2] Understanding the stability of a dynamical system is of utmost importance in situations where lives and property depend on the system's behavior. Indeed, Chen states that the "[s]tability of a dynamical system, with or without control and disturbance inputs, is a fundamental requirement for its practical value, particularly in most real-world applications."[3]

The first four models of the Boeing F/A-18 contained a mode in their flight control laws that demonstrates the implications of incomplete knowledge of nonlinear dynamical systems. This mode is highly unstable and results in violent oscillations around the roll and yaw axes as the aircraft quickly loses altitude. The mode has been called the "falling-leaf" mode due to the similarity between the aircraft's behavior and the motion of a leaf fluttering to the ground.[4] Though no lives have been claimed by this unstable phenomenon, several aircraft were lost when their pilots ejected after triggering this mode. Due to the nature of nonlinear dynamical systems, the designers of the flight control law simply had no knowledge that the falling-leaf mode existed.

II. Introduction

The previous example demonstrates the importance of understanding stable and unstable regions of a nonlinear dynamical system. One way to achieve this is by *linearizing* the nonlinear system about an equilibrium point. However, this method is only valid for regions close to the equilibrium point. Additionally, there is a loss of accuracy due to the linearization process. Therefore, linear analysis is entirely inadequate to understand highly nonlinear phenomena, such as the falling-leaf mode.

A framework has been developed to determine the long-term behavior of a nonlinear dynamical system without large approximations or restrictions on the domain of application. This framework uses concepts from the field of statistical physics. Specifically, the dynamical system is represented as a matrix called a *stochastic* or *Markov* matrix. Once this matrix has been created, future states of the system can be predicted through multiplication by the stochastic matrix, a linear operation.

The primary advantage of this probabilistic framework should now be evident. Even though the stochastic matrix contains a form of the nonlinear dynamics used to create it, the matrix itself becomes a linear operator. Thus, eigenvalue and eigenvector analysis can be applied to the stochastic matrix. In essence, this framework creates a linear representation of the nonlinear system without any loss of fidelity.

III. Theory: Markov Chains

A brief background on the theory behind the probabilistic framework will now be presented. The basis of the approach is a *Markov chain*.[5]

A Markov chain is a probabilistic way to predict future events that depend only on the current state and not the sequence of states that preceded it. This is done through a stochastic matrix M, also called a Markov matrix, that is populated according the following rule: The entry in matrix M with indices i and j is the probability that the system is in state j and reaches state i in the next time step:

$$M_{ij} = P(j \to i) \tag{1}$$

The states i and j can represent different things depending on the problem. For instance, they might represent different cities and $P(j \to i)$ would mean the probability that an individual moves from city j to city i.

It can be seen that M will be a square matrix with dimension equal to the number of independent states. Additionally, the sum of the entries of each column of M will be 1. Once the stochastic matrix has been constructed, it contains a probabilistic representation of the system used to generate it. Future behavior can be determined simply by multiplication by the stochastic matrix, which is a linear operation:

$$p^+ = Mp \tag{2}$$

In this formula, p is a column vector whose entries represent the probability of being in a given state. The sum of the entries of p must equal 1. The column vector is multiplied by the stochastic matrix, yielding a new probability vector, p^+ that represents the probability distribution after one time step. Successive applications of the stochastic matrix can be applied to continue stepping through time. If enough iterations are applied, the system will eventually approach a stable distribution, where the probability vector does not change under application of the stochastic matrix:

$$p^* = Mp^* \tag{3}$$

Thus, p^* is the eigenvector of M corresponding to an eigenvalue of 1. This eigenvector is sometimes called the *stationary probability vector*. To find the long-term behavior of a Markov chain, it is sufficient to find this eigenvector. It is noted that p^* depends only on the stochastic matrix. This means that the long-term probability of being in a given state is independent of the initial state.

IV. Generating the Stochastic Matrix

With this probabilistic framework, the problem of determining the stable region of a dynamical system has been reduced to an eigenvector problem on the stochastic matrix, M. Therefore, a means of computing this matrix must be derived.

Since the Markov chain requires a finite number of possible states, the domain of the problem must be discretized. This accomplished using a rectangular grid. This is by no means the only way to discretize the domain; it was selected due to its ease of implementation.

Now, an interpretation of the probability in Equation 1 can be made. States i and j are taken to represent individual grid cells. Therefore, $P(j \rightarrow i)$ represents the probability that the system's state is in cell j and ends up in cell i after some timestep.

The Markovian probability can be computed in the following way. The concepts will be introduced in two dimensions and then generalized to higher dimensions.[a] First, grid cells j and i are represented as a *polygon*. The nonlinear dynamics are applied to each of the *four* vertices of the polygon, creating a new, transformed polygon, j^t. The *area* of the intersection of polygon j^t with polygon i divided by the area of j is the Markovian probability:

$$M_{ij} = P(j \rightarrow i) = \frac{\text{Area}(j^t \cap i)}{\text{Area}(j)} \tag{4}$$

Figure 1 gives a graphical interpretation for the two-dimensional case. Cells i and j have been labeled, and the transformed vertices belonging to j^t can be seen. The shaded region represents the intersection between polygons j^t and i.

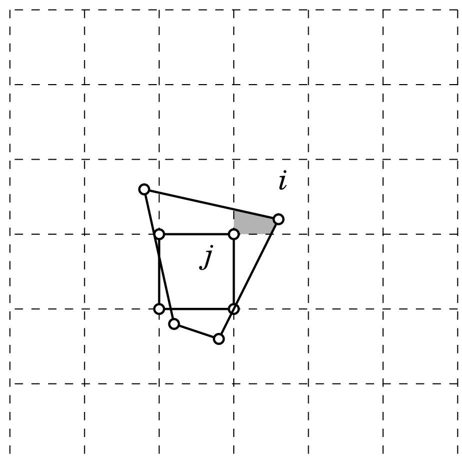

Figure 1: Definition of the Markovian probabilities in two dimensions.

Though the concept was developed in two-dimensions, the process is valid in N dimensions, with the following corrections: Instead of *polygons*, the N dimensional objects are called *polytopes*.[6] The number vertices of a hyperrectangular polytope is 2^N. Finally, *area* is replaced by the N-dimensional concept of *volume*.

V. Results

The Van der Pol oscillator was used as a nonlinear test case for the probabilistic framework[b]. This system contains a stable region called a *limit cycle*; in the long term, all trajectories converge upon this region.

The goal was to use the framework to determine the limit cycle. This required the creation of the stochastic matrix using the process described above. Next, standard eigenvector analysis was conducted on the stochastic matrix to produce the stationary probability vector. Grid cells with a high stationary probability define the limit cycle. When these cells are plotted, the result is a graphical representation of the limit cycle.

Figure 2 demonstrates the computed limit cycles. The solid white line defines the true limit cycle. The grey region gives the limit cycle as determined by the probabilistic algorithm. Starting with 2a, the limit cycle is computed quickly at a low resolution. Each successive iteration requires more execution time, but doubles the resolution. After six iterations, the computed limit cycle is quite well-defined.

At each level of resolution, the true limit cycle is contained within the probabilistic representation. This is important because even at the lowest resolution, the framework produces the correct result and yields

[a] The italicized terms pertain solely to two dimensions

[b] See the Appendix for more information on the Van der Pol oscillator

valuable information. As the resolution increases, the set is further refined until it represents the true limit cycle very closely.

VI. Discussion

The goal of this research was to develop a new framework for analyzing nonlinear dynamical systems. To that end, a technique from the field of statistical physics was adapted. Specifically, a stochastic matrix is generated for the system. Though the matrix contains a representation of the nonlinear nature of the system, it is itself a linear operator, allowing for eigenvector and eigenvalue analysis. It is subsequently straightforward to find the stable and unstable regions.

The framework was applied to the Van der Pol oscillator, demonstrating that it generates the correct limit cycle at different resolutions.

This technique could have important applications wherever the behavior of nonlinear dynamical systems is of critical importance. For instance, designers of flight control laws could run simulations to detect unstable regions like were present in the F/A-18. This would allow appropriate corrections to be applied before the aircraft reached production. The result is a sharp increase in the safety and robustness of the flight controllers.

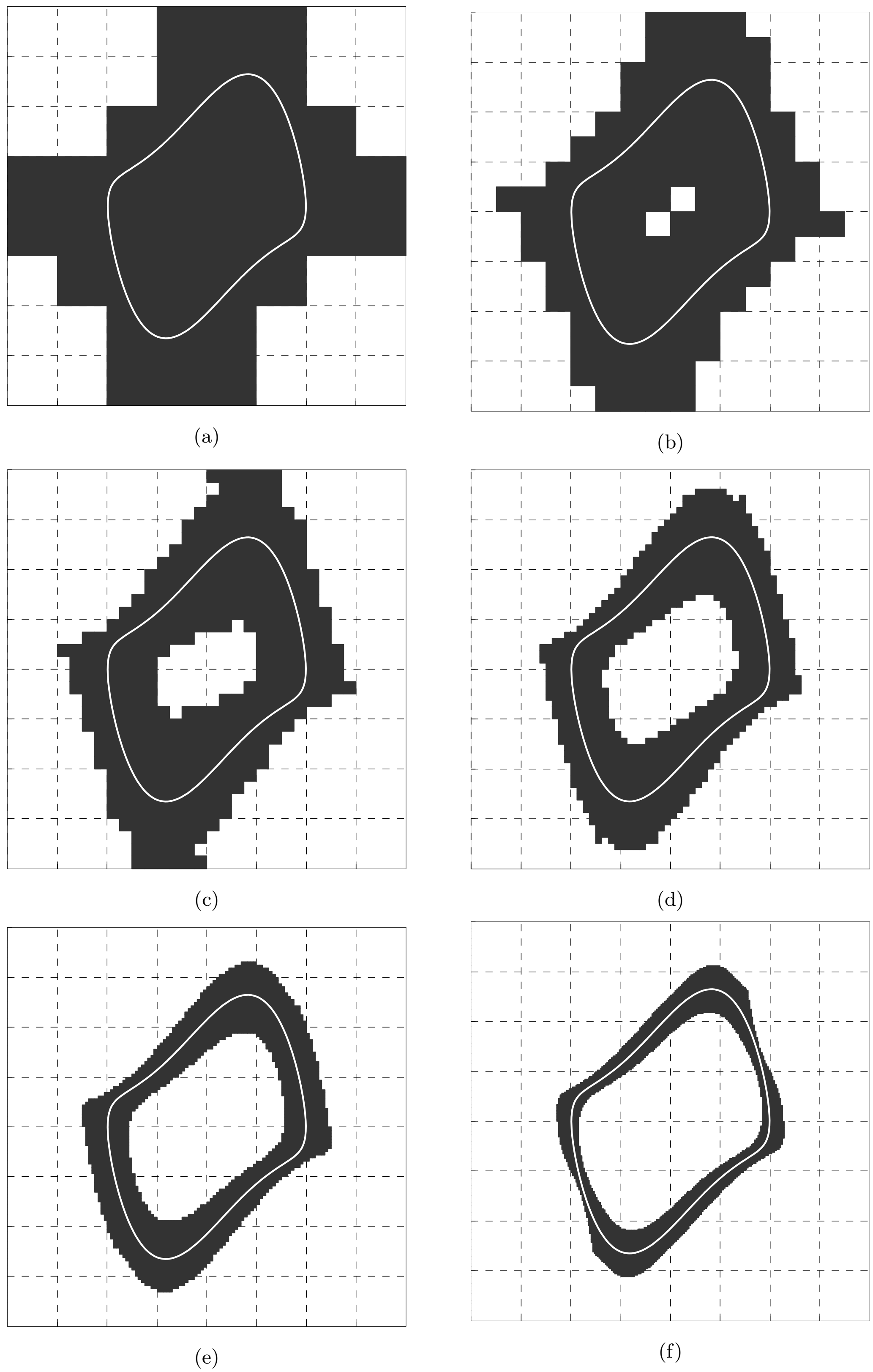

Figure 2: Computation of Van der Pol oscillator limit cycle at increasing resolutions. Solid white line indicates limit cycle.

Appendix

Van der Pol Oscillator

The Van der Pol oscillator was used as an example nonlinear system when describing the implementation of the probabilistic framework for understanding nonlinear dynamical systems. Equation (5) gives the second-order differential form of the system. This system models stable oscillations in electrical circuits with vacuum tubes. It has also been used to model neurological functions[7] and geologic faults.[8]

$$\frac{\partial^2 x}{\partial t^2} - \epsilon(1 - x^2)\frac{\partial x}{\partial t} + x = 0 \tag{5}$$

As with any second-order differential equation, the Van der Pol equation can be transformed into a system of two first-order equations. In this way, the system can be either two-dimensional or three-dimensional: the parameter ϵ can be fixed, in which case, the system will be two-dimensional, or it can be left as an additional independent variable, causing the system to take on a third dimension.

The Van der Pol oscillator contains a stable region known as a limit cycle. A limit cycle is a region of attraction toward which trajectories converge. Once in the limit cycle, the trajectories are periodic. In the case of the Van der Pol oscillator, all trajectories converge to the limit cycle regardless of initial condition. This makes the system ideal to use as a test case. Figure 3 shows the limit cycle of the system when $\epsilon = 1$.

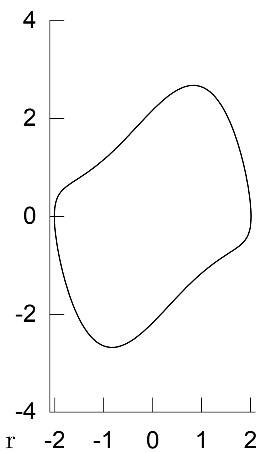

Figure 3: Limit cycle of Van der Pol oscillator when $\epsilon = 1$.

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