1. Introduction

A new method for the solution of statistical problems describing the motion of ensembles of particles is discussed. The method is an alternative to the usually used Euler algorithms.

The method can be reduced to transformation of statistical distributions, beginning from distributions described by generalized functions by Dirac. The Dirac function was used in orbital problems in [1, 2, 3-6, 7-12].

Analytical formulas for the steady flows of orbital media and interplanetary dust particles in the vicinity of the Earth, considering the shadowing by the Earth, are obtained. The numerical solution of the unsteady flow in vicinity of a gravitating center is studied. The graph of the unsteady drag force acting on a body at the dust region is depicted.

2. Space density induced by a fixed particle

Let’s have a look at the proposed technique. In the geocentric frame (ϕ, ω, r), where ϕ is a latitude, ω is a longitude, r is a radius of a point, the density induced by a particle can be written in the following way:

\[ \rho_1(\phi, \omega, r) = \delta(\sin \Phi - \sin \phi) \cdot \delta(\Omega - \omega) \cdot \delta\left(\frac{R^3}{3} - \frac{r^3}{3}\right), \]  

(1)

where Φ, Ω, and R is a latitude, a longitude and a radius of the particle. This expression meets normalization requirements.

Firstly,

\[ \int_0^{2\pi} d\phi \int_{-\pi/2}^{\pi/2} \cos \phi d\phi \int_0^\infty r^2 dr \times \rho(\phi, \omega, r) = 1, \]

i.e. there is one particle in the space.

And then,

\[ \int_0^{2\pi} d\Omega \cdot \int_0^{\pi/2} \cos \phi d\phi \cdot \int_0^\infty R^2 dR \times \rho(\phi, \omega, r, \Phi, \Omega, R) = 1, \]

i.e. the particle exists somewhere in the space.

3. Space density induced by a particle averaged over an orbit

The equation (1) describes a momentary density distribution. If the particle is moving then \( \Phi(\Phi(t), \Omega(\Omega(t)), \) and \( R = R(t), \) and the density depends on time.

Further, the procedure of averaging is used essentially.

Let the particle move in a closed orbit. Then the average density is determined by the following integral:

\[ \rho_2(\phi, \omega, r) = \frac{1}{T} \int_0^T \rho_1(\phi, \omega, r, t) dt, \]  

(2)
where \( T_i = \frac{2\pi}{\sqrt{(1-e^2)^3/\mu_0}} \) is the period of an orbit, 

\( p \) is a focal parameter.

Using properties of the delta-function, \( \delta(\Omega-\omega) \), one gets the following expression for the density of distribution of a particle averaged over its orbit:

\[
\rho_2(\varphi, \omega, r) = \frac{r^2(1-e^2)^3}{2\pi p^2} \delta (\sin \Phi - \sin \varphi) \delta\left( \frac{R^3}{3} - \frac{r^3}{3} \right) \frac{\cos \varphi}{1 - \sin^2 i \cos^2 (\omega - \Omega_0)},
\]

(3)

The functions \( \Phi=\Phi(\omega) \) and \( R=R(\omega) \) are determined by the following relations:

\[
\sin \Phi = \frac{\sin i \sin (\omega - \Omega_0)}{\sqrt{1 - \sin^2 i \cos^2 (\omega - \Omega_0)}},
\]

(4)

\[
\cos \Phi = \frac{\cos i}{\sqrt{1 - \sin^2 i \cos^2 (\omega - \Omega_0)}},
\]

(5)

\[
R = \frac{p}{1 + e \cos \theta},
\]

(6)

\[
\cos (\theta - \theta_0) = \frac{\cos i \cos (\omega - \Omega_0)}{\sqrt{1 - \sin^2 i \cos^2 (\omega - \Omega_0)}},
\]

(7)

where \( \Omega_0 \) and \( \theta_0 \) are a longitude of ascending node and an argument of pericentre of an orbit.

4. Space density induced by a particle averaged over a surface

Let the ascending node precess. Then longitude \( \Omega = \Omega_0(t) \), and the equation (3) determines the function \( \rho_2(\varphi, \omega, R, t) \) dependent on time. Evidently, the longitude of the ascending node is distributed uniformly over \([0, 2\pi]\). The particle is averaged over a surface of rotation, and the associated averaged space density is described by the following integral

\[
\rho_3(\varphi, \omega, r) = \frac{1}{T_2} \int_0^{T_2} \rho_2(\varphi, \omega, r, t) dt,
\]

(8)

where \( T_2 \) is a period of the precession.

Considering the relation between the period of the precession and the speed of motion of the ascending node \( T_2 = \frac{2\pi}{d\Omega_0/dt} \), and considering that each value \( \sin \Phi \) is repeated twice a period, one obtains

\[
\rho_3(\varphi, \omega, r) = \frac{r^2(1-e^2)^3}{2\pi p^2} \left[ \delta (\frac{R^3}{3} - \frac{r^3}{3}) + \delta (\frac{R^3}{3} - \frac{r^3}{3}) \right] \frac{1}{\sqrt{\sin^2 i - \sin^2 \varphi}},
\]

(9)

where \(-i \leq \varphi \leq i\),

\[
R_1 = \frac{p}{1 + e \cos \theta}, \quad R_2 = \frac{p}{1 - e \cos \theta}, \quad \cos (\theta - \theta_0) = \frac{\sqrt{\sin^2 i - \sin^2 \varphi}}{\sin i}.
\]

5. Full average of the space density induced by a particle

Really, the pericentre precesses, too. Then the argument of pericentre, \( \theta = \theta_0(t) \), and there is an averaging over a body of rotation determined by conditions \( r_1 \leq r \leq r_2 \), \(-i \leq \varphi \leq i\), where \( i - \)
the inclination, and \( r_1 \) and \( r_2 \) – pericentre and apocentre of the orbit. If this precession is uniform, then the average density is determined as

\[
\rho_4(\varphi, \omega, r) = \frac{1}{T_3} \int_0^{T_3} \rho_3(\varphi, \omega, r, t) \, dt,
\]

where \( T_3 = \frac{2\pi}{d\theta_0/dt} \) is the period of the apsidal precession.

Considering that each value of the radius, \( r_1 < r < r_2 \), is repeated twice an orbit (at ascending and descending branches of an orbit), one gets

\[
\rho_4(\varphi, \omega, r) = \frac{1}{2\pi^3} \frac{1}{\sqrt{\sin^2 \varphi - \sin^2 \varphi}} \frac{(1 - e^2)^{3/2}}{p r^2 \sqrt{e^2 - \left(\frac{p}{r} - 1\right)^2}},
\]

or in symmetrical form:

\[
\rho_4(\varphi, \omega, r) = \frac{1}{2\pi^3 a r} \frac{1}{\sqrt{\sin^2 \varphi - \sin^2 \varphi} \cdot \sqrt{(r - \eta)(r_2 - r)}},
\]

where \(-i \leq \varphi \leq i\), \( r_1 \leq r \leq r_2 \).

This expression is known as the formula by D. Kessler.

Note that all above derived distributions meet the normalization requirement:

\[
\int_0^{2\pi} \int_{-\pi/2}^{\pi/2} \cos \varphi r dr d\varphi \int_0^\infty r^2 dr \times \rho(\varphi, \omega, r) = 1,
\]

and can be used in many problems.

6. Nondimensional parameters in the steady problem

Below, the flows of dust particles in the vicinity of a gravitating center are studied when the motion of the particles is hyperbolic. Two free parameters appear: velocity of the particles in infinity and an impact parameter. For generality, lets use the following dimensionless parameters:

\[
\tilde{v} = \frac{v}{v_0} \quad \text{is a velocity of the particle,}
\]

\[
\tilde{r} = \frac{r \cdot v_0^2}{\mu} \quad \text{is a radius-vector,}
\]

\[
\tilde{x} = \frac{x \cdot v_0^2}{\mu} \quad \text{is an impact parameter,}
\]

\[
\tilde{\rho} = \frac{\rho}{\rho_0} \quad \text{is a space density,}
\]

where \( v_0 \) and \( \rho_0 \) are the velocity and space density at infinity, \( \mu \) is a gravitation constant of the Earth, \( \mu = M \cdot G \), \( M \) is the mass of a gravitating center, \( G \) is the universal gravitation constant.

Furthermore, only dimensionless parameters are used and the sign of nondimensionalization is omitted.

7. The problem on the motion of a unidirectional flow of meteoroids in vicinity of the Earth
This steady hyperbolic problem describes the motion of seasonal meteoroid streams.
So, there is a parallel flow of particles at infinity. The appropriate initial dimensionless velocity equals 1 (dimensional velocity is \( v_0 \) as shown the Fig.1). The flow is incident on a gravitating center.

Fig.1. A problem on a meteoroid flow in the vicinity of the Earth.

Suppose that a particle uniformly distributed over azimuthal angle. The space density induced at one instant \( t \) can be described by the formula

\[
\rho(r, \theta, t) = \frac{1}{2\pi \cdot R^2(t) \cdot \sin \Theta(t)} \delta(R(t) - r) \cdot \delta(\Theta(t) - \theta), \tag{14}
\]

where \( R(t) \) and \( \Theta(t) \) are the functions that describe the radial and angular motion of the particle.

The normalization requirement is as follows

\[
\rho(r, \theta, t) = \int_0^\infty \int_0^{2\pi} \rho(r, \theta, t) \cdot 2\pi \cdot r^2 \sin \theta \cdot dr \cdot d\theta = 1. \tag{15}
\]

The average density induced by the particle for a time interval \( T \) is

\[
\bar{\rho}(r, \theta) = \frac{1}{T} \int_0^T \rho(r, \theta, t) \cdot dt. \tag{16}
\]

Let’s do integration using one of the delta-functions in (14) and the relation for angular motion \( \left| \frac{d\Theta}{dt} \right| = \frac{|v_\theta|}{r} \). Consider also that the equation of motion \( \Theta(t) - \theta = 0 \) has two solutions for each radius \( r \). The last leads to two values of impact parameters for particles which pass through each space point \( (r, \theta, \varphi) \):

\[
\chi_{1,2} = \frac{1}{2} \left( r \sin \theta \pm \sqrt{r^2 \sin^2 \theta + 4r(1 - \cos \theta)} \right). \tag{17}
\]

The streams with positive impact parameters can be called as direct streams, and the streams with negative impact parameters can be called scattered streams, because the particles get to the point rounding the gravitating center. The total density is determined by the sum of these two streams:

\[
\bar{\rho}(r, \theta) = \frac{1}{T} \sum_{i=1}^{2} \frac{1}{2\pi \cdot R^2(\Theta) \cdot \sin \theta} \cdot \delta(R(\Theta) - r) \cdot \frac{1}{\left| \frac{v_\theta}{R(\Theta)} \right|}. \tag{18}
\]
The function $R(\theta)$ reflects a dependence on the angle of radius-vector of the moving particle and it is determined through the above mentioned time dependence $R(t)$.

The following relation derives from the law of conservation of angular momentum:

\[
\frac{\varrho}{r} = \frac{\chi}{R^2(\theta)}.
\]  

Then the integration over the impact parameter must be fulfilled. If the flow rate at infinity is $f_0$, then the number of particles in an elementary step over the impact parameter is

\[
N = 2\pi \chi \cdot d\chi \cdot T \cdot f_0,
\]

Then the total space density of the dust is

\[
\bar{\rho}(r, \theta) = f_0 \sum_{i=1}^{\infty} \int_{0}^{\pi} \frac{\delta(R(\theta) - r)}{\sin \theta} \cdot d\chi.
\]

The integrals in (21) can be solved using delta-function’s properties. The final space density is

\[
\bar{\rho}(r, \theta) = f_0 \sum_{i=1}^{\infty} \frac{1}{\sin \theta} \left| R'_\chi \right|
\]

where

\[
R'_\chi = \frac{dR}{d\chi} = 2 \frac{r}{\chi} - \frac{r^2}{\chi^2} \sin \theta
\]

The equation (22) determines the quasi steady density of particles in the vicinity of the Earth, when the Earth crosses a meteoroid stream.

The other solution of this problem was developed by Jones & Poole [13] based on the Euler method.

8. Features of the focusing effect and shadowing by the Earth

The important feature of the dust flow in vicinity of a gravitating center is a sharp focusing at the axis behind the center (Fig. 2).

Fig.2. Homogeneous meteoroid stream with velocity of 10 km/sec in gravitation field of the Earth. The sketch is scaled.

The bend effect can essentially reduce the region of the Earth shadow. Contrariwise, beginning from some critical distance there is a sharp increase of the intensity of the fluxes. Behind the Earth for distances greater than a critical distance, at the axis of the stream there is a singularity in the dust density described by the expression

\[
\rho(r, \theta) \approx f_0 \cdot \frac{\sqrt{2}}{|\pi - \theta| \cdot \sqrt{r}}.
\]
The singularity has place for any speed of meteoroids, but when the speed is sufficiently large, the critical distance is far enough from the Earth. The critical distance equals \( r_s = \frac{\chi^2}{2} \). For the geostationary orbit one has \( r_{GSO} \approx R_{Em} \approx 6.52 \), where \( R_{Em} \) is a minimal particle-Earth distance of convergence that equals the radius of the Earth plus the height of the atmosphere (Table 1). If the critical distance appears at the geostationary region, then \( r_{GSO} = \frac{R_{Em}(R_{Em} + 2)}{2} \) and \( R_{Em} \approx 11 \). It corresponds to meteoroid speed about 26 km/sec.

For greater speed the edge is further from the Earth. But in this approach the singularity has a place for any speed of the meteoroids stream. The accurate calculation of the risk of meteoroid impacts needs selection of such singularities. It is a real danger. And weak meteoroid streams, that are not selected now, can create enough dangerous density along some directions. So the structure of the meteoroid streams and their evolution must be studied accurately.

### Table 1. Dimensionless radius of atmosphere of the Earth.

<table>
<thead>
<tr>
<th>( v ), km/sec</th>
<th>5</th>
<th>10</th>
<th>15</th>
<th>20</th>
<th>25</th>
<th>30</th>
<th>40</th>
<th>50</th>
<th>70</th>
</tr>
</thead>
<tbody>
<tr>
<td>( R_{Em} )</td>
<td>0.41</td>
<td>1.64</td>
<td>3.68</td>
<td>6.54</td>
<td>10.2</td>
<td>14.7</td>
<td>26.2</td>
<td>40.9</td>
<td>80.2</td>
</tr>
</tbody>
</table>

The characteristic density of meteoroid distributions near the Earth can be evaluated using the formula for density of dust distribution in the transverse plane that passes through the Earth’s center (the main plane). The density is as following (see also Fig. 3):

\[
\rho(r) = f_0 \cdot \left( 1 - \frac{1}{16} \cdot \frac{1}{R^2} \left( \frac{1 + \sqrt{1 + \frac{4}{R}}}{4} \right)^4 \right) \cdot \left( \frac{1}{16} - \frac{1}{R^2} \left( \frac{1 - \sqrt{1 + \frac{4}{R}}}{4} \right)^4 \right).
\]  

(25)

Fig.3. The profile of density distribution in the transverse plane (the attenuation is negligible).

The radius-vector of the point where a meteoroid crosses the transverse plane is \( r_e = \frac{\chi^2}{1 + \chi} \). The radius tends to the impact parameter with an increase in this parameter.
9. On isotropic flows

The background, sporadic meteoroids have a rather isotropic distribution [14, 15]. The simple method of estimation was proposed in [16] (without shadowing). The principle of symmetry and the conservation law, which allows to have simple and reliable estimates for meteoroid space density versus radius-vector in the near-Earth region, were used. Using the delta-function the following expression for the dust density can be derived (like [16], without shadowing):

$$
\rho(r) = \rho_0 \sqrt{1 + \frac{2}{r}} .
$$

(26)

The shadowing plays an important role in the distribution of the meteoroid dust. There is a critical radius $R_f$

$$
R_f = \frac{R_{Em}(R_{Em} + 2)}{2}.
$$

If $r < R_f$, then only the direct flux comes to the point and

$$
\rho(r) = \rho_0 \left[ \sqrt{1 + \frac{2}{r}} - \frac{1}{2} \sqrt{\left(1 + \cos \varphi \right) \left(1 + \frac{4}{r} \cos \varphi \right) + (1 - \cos \varphi)} \right],
$$

(28)

where $\varphi = \theta_0 + \theta_e$,

$$
\cos \theta_0 = -\frac{1}{e}, \quad \cos \theta_e = \left(\frac{R_{Em}(R_{Em} + 2)}{r} - 1\right) \frac{1}{e}, \quad \sin \theta_e = \frac{1}{e} \sqrt{e^2 - \left(\frac{R_{Em}(R_{Em} + 2)}{r} - 1\right)^2},
$$

$$
e = \sqrt{1 + R_{Em} \cdot (R_{Em} + 2)}.
$$

But, if $r \geq R_f$, then the direct flux and a part of the scattered flux come to the point and

$$
\rho(r) = \frac{\rho_0}{2} \left[ \sqrt{1 + \frac{2}{r}} + 1 \left(1 + \frac{1}{2} \left[1 + \cos \varphi \left(\frac{1 + \frac{4}{r} \cos \varphi}{r} - (\cos \varphi + 1)\right)\right]\right]\right]
$$

(29)

The different profiles of the dust density are shown in the Fig. 4.

Fig. 4. Density of the interplanetary dust near the Earth for initial speed of 15 km/sec. 1- isotropic flow (without shadowing), 2- isotropic flow
considering shadowing, 3 – the density in the central transverse plane for the case of a unidirectional stream.

So, the gravitational bending of particle trajectories leads to an essentially decreasing of the region where the shadowing effects must be considered, while the angular “dead zone” increases. In the Fig. 5 the angle $\gamma$ determines the solid angle that is free from meteoroid fluxes(without bending), and the $\gamma'$ is the same angle considering bending, $\gamma' \geq \gamma$.

![Fig. 5. Influence of the gravitational bend on the angular “dead zone”](image)

It should be indicated that these are only characteristic estimates. The accurate estimates must be done by numerical integration using the above given distributions for unidirectional flows, because the sporadic meteoroid population has an anisotropy too.

10. Cross-section of absorption

The cross-sections of absorption play an important role in the evolution of interplanetary dust streams which cross the planet orbit. For any meteoroid trajectory the point of minimal distance from a planet is determined by equation

$$\tan \theta_m = -\chi$$  \hspace{1cm} (30)

If the appropriate radius-vector is less than $R_{pm}$, such a particle is absorbed by the planet. This radius is determined by the planet and its atmosphere. The critical impact parameter is determined by the following relation (considering the signs):

$$r_m = \frac{\chi_m^2}{1 + \sqrt{1 + \chi_m^2}}$$  \hspace{1cm} (31)

Of course, if $\chi_m \gg 1$, then $r_m \approx \chi_m$.

Thus, the minimal impact parameter is determined by equation

$$\chi_m^2 = r_m^2 + 2r_m$$  \hspace{1cm} (32)

So, the apparent cross-section of absorption, $S = \pi \cdot \chi_m^2$, is described by the formula

$$S = \left(1 + \frac{2}{r_m}\right) S_0$$,  \hspace{1cm} (33)

where $S_0 = \pi \cdot r_m^2$ is a usual cross-section of absorption.

11. Unsteady gravitational drag in an unbounded dust media
The steady problem of gravitational drag was considered by Chandrasekhar [17-19]. There, an attempt is made to solve a simplified unsteady problem. Let the gravitating center initially be fixed in an unbounded dust media. The media is uniform (to some reasonable extent!). At $t=0$ the gravitating center gets a velocity taken as unity (Fig.6).

![Fig. 6. The problem of instant motion of a gravitation center in a dust media.](image)

The problem needs additional specific dimensionless parameters.

\[ \tilde{t} = \frac{t \cdot v_0^3}{\mu} \]  
\[ \tilde{F} = F \cdot \frac{v_0^2}{m \rho_0 \mu^2} \]

is an elapsed time.  

is a force acting on a gravitating center.

The motion of any dust particle can be determined by following two equations:

\[ t = f(\theta, \theta_0, r_0) \]
\[ r = g(\theta, \theta_0, r_0). \]

where $\theta_0$, $r_0$ are the polar coordinates of a dust particle at the initial time $t=0$, and $\theta$, $r$ are the coordinates at an arbitrary time $t$.

At any time the density of distribution of the dust is a sum of several components

\[ \rho(r, \theta, t) = \sum_i \rho_i(r, \theta, t), \]

where each of these components is described by the formula of common view:

\[ \rho_i(r, \theta, t) = \frac{r_0^2 \sin \theta_0}{r^2 \sin \theta} \frac{\partial f}{\partial \theta} \left( \frac{\partial g}{\partial \theta_0} - \frac{\partial g}{\partial r_0} \frac{\partial f}{\partial \theta_0} \right), \]

\[ r_0 = r_0^{(i)} \text{ and } \theta_0 = \theta_0^{(i)} \] are solutions of the system (34).

Note that each value on the right side of equation (36) has the index $i$ that is omitted.

There are several solutions of the equation (34) in the region of finite trajectories, but there are only two solutions in the region of infinite trajectories, that is most important in the problem. Thus $\rho_p(r, \theta, t)$ is a density of a direct flow and $\rho_d(r, \theta, t)$ is a density of a scattered flow. The density of the direct flow prevails everywhere, except the narrow wake region.

The detailed analysis of the expression (14) is rather complicated, and it needs additional verifications. Such verifications have been performed in different ways (Fig.7).
Firstly the limiting expression at $t \to \infty$ is obtained. Numerical calculations show identity of results.

The second way is investigation of temporary behavior. Fig. 7 shows the temporary increments of the dust density of the direct component when $\theta = 90^\circ$, $r = 10$ and $r = 30$.

Integration of the gravitational drag force over the space gives the following graphic (Fig. 9).

The drag force acting on the gravitating center that moves in an infinite media grows with time unlimitedly, and the density increment at great distances contribute the major input into the drag force.

**Conclusions**

A method of generalized functions for the solution of problems of statistical celestial mechanics is proposed. The analytical solutions of several problems of a steady periodic and aperiodic flow in the vicinity of a gravitating center are obtained. The problems of meteoroid distribution in the near-Earth space were studied, considering focusing and shadowing effects. The problem of unsteady motion of gravitating body in a dust environment is investigated. The drag force via time is given.

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References