RADAR-OPTICAL OBSERVATION MIX

Felix R. Hoots

Deep space satellites, having a period greater than or equal to 225 minutes, can be tracked by either radar or optical sensors. However, in the US Space Surveillance Network only a limited amount of radar tracking resources is available. Therefore, it would be useful to have a quantitative way to decide how to best distribute radar and optical resources to optimize the resulting orbit prediction accuracy. The covariance provides a generally accepted way to assess prediction quality. For circular satellite motion synchronized with the Earth motion, we are able to formulate the covariance analytically. The analytical expressions reveal several interesting properties of the covariance. The formulation allows determination of the optimum choice of resource use between radar and optical measurements. It also reveals that the optimum mix depends not only on sensor quality but also on fit span length. Finally, the analytic formulation unveils a curious and surprising link between the orbit determination corrections and where each measurement error occurs in time.

INTRODUCTION

A deep space satellite is defined to be a satellite having an orbital period greater than or equal to 225 minutes. This includes three main categories of orbits: synchronous, semi-synchronous circular, and semi-synchronous highly eccentric. This definition was developed by the Air Force in 1977 based on two factors.

- Altitude where the perturbation effect of lunar-solar gravity is approximately equal to that of the first Earth zonal harmonic
- Altitude upper limit for most space surveillance radars at that time

Today the US Space Surveillance Network includes dish type radars that can track deep space satellites at synchronous altitudes. However, there are only three such radars and tracking resources are limited. Thus, the majority of deep space satellite observations come from powerful telescopes with CCD focal planes. Since radar resources are limited, a natural question is how one should best employ these limited resources. Miller1 has looked at this problem using a numerical approach. This paper addresses the problem with an analytic method.

1 Senior Engineering Specialist, The Aerospace Corporation, 15049 Conference Center Drive, Chantilly, VA 20151
The resulting analytic formulation allows us to examine the functional dependence of prediction accuracy on such variables as track length, number of observations, measurement accuracy, and mix of observation types. Through the use of such formulas, analysts can then properly focus detailed numerical trades in a much more efficient manner.

**APPROACH**

In order to compare radar and optical sensors, we can think of the accuracy of each component of the measurement, whether range or angle, as an equivalent distance accuracy. Because of the way a radar works, the range tends to be the most accurate measurement while the angles are not nearly as good. In contrast, optical sensors have angular measurements of quality comparable to radar range accuracy, but they only provide a two dimensional observation.

In order to examine what mix of radar and optical resources will be best, we will assume that the covariance will provide an assessment of the state vector quality resulting from an orbit determination using a given mix of radar and optical observations. The observation mix that minimizes the covariance will be judged to be the “best”.

With a few simplifying assumptions, we can formulate the problem analytically. Much greater insight into the behavior of the covariance can be gained by looking at a closed form solution than can ever be perceived from countless numerical results.

**PROBLEM MODEL**

The physical forces acting on a deep space satellite are well known and can be modeled extremely well using numerical integration. Thus, the main source of orbit determination and prediction error for deep space satellites is measurement noise and sparseness of data, not physical modeling errors. Since a shortfall in modeling is not the issue, we can use a two body model for both the truth reference and the trajectory model and simply examine the error induced by measurement noise.

In order to develop a simple model for our problem, we will restrict our analysis to circular orbits. This will accommodate both the synchronous and semi-synchronous circular deep space orbits, but will not address semi-synchronous highly eccentric orbits. We will address this type orbit in a future paper.

Since the two body motion is confined to a plane, we formulate our analysis in the plane of motion. The motion can be described by
where

\[ a_0 = \text{epoch value of semimajor axis} \]
\[ M_0 = \text{epoch value of mean anomaly} \]
\[ n_0 = \text{epoch value of mean motion} \]
\[ t = \text{time since epoch} \]

Since we are restricting our analysis to satellites whose motion is synchronized (or semi-synchronized) with the Earth, we assume that a sensor can observe the satellite each time the sensor moves through the orbital plane. Because of the synchronization, we assume the measurements are taken when the satellite is directly overhead and the measurements occur in the plane of motion of the satellite.

Assume that the radar takes regular measurements in the radial and angular direction. Figure 1 illustrates the coordinate frames when measurements of a synchronous satellite are taken. The coordinate system has been selected so that both the sensor angle measurement, \( \theta \), and the mean anomaly, \( M \), of the satellite are measured from a common line, the inertial x axis. For a semi-synchronous satellite, the sensor will have two observation opportunities each day, but the sensor measurement angle is still reckoned from the x axis.

Figure 1: Sensor and Satellite Coordinates
The radar measurements will not be perfect, but rather can be characterized by their standard deviations, $\sigma_{\rho_j}$ and $\sigma_{\theta_j}$, respectively. Further, assume there is an optical system that takes observations each time the satellite is directly overhead. Assume that the optical system takes regular measurements in angles only, with errors being characterized by their standard deviation, $\sigma_{\alpha_j}$. The development that follows does not depend on the errors being normally distributed, but does assume zero mean error.

The jth radar measurement can be represented as

$$\rho_j = \rho_{ij} + \sigma_{\rho_j}$$
$$\theta_j = \theta_{ij} + \sigma_{\theta_j}$$

and the jth optical measurement can be represented by

$$\alpha_j = \alpha_{ij} + \sigma_{\alpha_j}$$

where

$\rho_{ij} = a_0 - R_j$, true range for jth observation
$\sigma_{\rho_j} = $ radar range noise for jth observation
$\theta_{ij} = M_j + \eta_j$, true angle for jth observation
$\sigma_{\theta_j} = $ radar angle noise for jth observation
$\alpha_{ij} = \theta_{ij}$, true optical angle for jth observation
$\sigma_{\alpha_j} = $ optical angle noise for the jth observation

**ORBIT DETERMINATION**

The differences between the measurements and the estimate of the satellite orbit are described by

$$\Delta\rho = \left(\frac{\partial \rho}{\partial M}\right) M_0 + \left(\frac{\partial \rho}{\partial n}\right) n_0$$
$$\Delta\theta = \left(\frac{\partial \theta}{\partial M}\right) M_0 + \left(\frac{\partial \theta}{\partial n}\right) n_0$$
$$\Delta\alpha = \left(\frac{\partial \alpha}{\partial M}\right) M_0 + \left(\frac{\partial \alpha}{\partial n}\right) n_0$$
where the subscript 0 indicates evaluation at the initial estimates of $n_o$ and $M_o$. In matrix form Eq. (2) is

$$B = A\Delta X$$

where

$$A = \begin{pmatrix}
\frac{\partial \rho}{\partial \rho_0} & \frac{\partial \rho}{\partial \theta_0} \\
\frac{\partial \theta}{\partial \rho_0} & \frac{\partial \theta}{\partial \theta_0} \\
\frac{\partial \alpha}{\partial \rho_0} & \frac{\partial \alpha}{\partial \theta_0}
\end{pmatrix} = \begin{pmatrix}
0 & H \\
1 & t \\
1 & t
\end{pmatrix}$$

$$H = \frac{2a_n}{3n_o}$$

$$B = \begin{pmatrix}
\Delta \rho \\
\Delta \theta \\
\Delta \alpha
\end{pmatrix}$$

$$\Delta X = \begin{pmatrix}
\Delta M_0 \\
\Delta n_o
\end{pmatrix}$$

If we multiply both sides of Eq. (3) by the weight matrix, we obtain

$$WB = W A \Delta X$$

where

$$W = \begin{pmatrix}
1/\sigma_\rho^2 & 0 & 0 \\
0 & 1/\sigma_\theta^2 & 0 \\
0 & 0 & 1/\sigma_\alpha^2
\end{pmatrix}$$

with $\sigma_\rho$, $\sigma_\theta$, $\sigma_\alpha$ being the standard deviations of the radar range and angle and optical angle measurements, respectively. Next we multiply Eq. (5) by the transpose of the $A$ matrix to get

$$A^T WB = A^T W A \Delta X$$
Assume there are a total of \( k + m \) observations spread over the time interval \( T \). Then take the summation over all observations to get

\[
\sum_{j=1}^{k+m} A^T WB = \left( \sum_{j=1}^{k+m} A^T WA \right) \Delta X \quad (7)
\]

Let us assume there are \( m \) observations from the optical system and \( k \) observations from the radar system. For the optical system we have

\[
\sum_{j=1}^{m} A^T WB = \frac{1}{\sigma_a^2} \left( \sum_{j=1}^{m} \Delta \alpha_j \right) \quad (8)
\]

and for the radar system we have

\[
\sum_{j=1}^{k} A^T WB = \frac{1}{\sigma_\rho^2} \left( \sum_{j=1}^{k} \Delta \theta_j \right) \quad (9)
\]

where

\[
G = \frac{\sigma_\rho^2}{\sigma_a^2}
\]

Let us assume that each of the observation types is spread uniformly over the time interval \( T \). Then for the optical system we have

\[
\sum_{j=1}^{m} A^T WB = \frac{1}{\sigma_a^2} \left( \Delta \sum_{j=1}^{m} (j-1) \sigma_a \right) = \frac{1}{\sigma_a^2} \left( \sum_{j=1}^{m} \sigma_a \right) \quad (8)
\]

and for the radar system we have
We introduce the following shorthand notation.

\[ P = \sum_{j=1}^{m} \sigma_{a_j} \]
\[ P_T = \sum_{j=1}^{m} (j-1) \sigma_{a_j} \]
\[ Q = \sum_{j=1}^{k} \sigma_{\theta_j} \]
\[ Q_T = \sum_{j=1}^{k} (j-1) \sigma_{\theta_j} \]
\[ R = \sum_{j=1}^{k} \sigma_{\rho_j} \]

Then

\[
\sum_{j=1}^{k} A^T WB = \frac{1}{\sigma_{\theta}^2} \left( \sum_{j=1}^{k} \sigma_{\theta_j} \right) \left( \sum_{j=1}^{k} \sigma_{\theta_j} \right) + \frac{1}{\sigma_{\rho}^2} \left( \sum_{j=1}^{k} \sigma_{\rho_j} \right) \left( \sum_{j=1}^{k} \sigma_{\rho_j} \right) + \frac{1}{\sigma_{\theta}^2} \left( \sum_{j=1}^{k} \sigma_{\theta_j} \right) \left( \sum_{j=1}^{k} \sigma_{\theta_j} \right) \left( \sum_{j=1}^{k} \sigma_{\theta_j} \right) + \frac{1}{\sigma_{\rho}^2} \left( \sum_{j=1}^{k} \sigma_{\rho_j} \right) \left( \sum_{j=1}^{k} \sigma_{\rho_j} \right) \left( \sum_{j=1}^{k} \sigma_{\rho_j} \right)
\]

Let us look at the right hand side of Eq. (7). For the optical system we have

\[
\sum_{j=1}^{m} A^T WA = \frac{1}{\sigma_{a}^2} \left( \sum_{j=1}^{m} t_j \right) \left( \sum_{j=1}^{m} t_j \right) + \frac{1}{\sigma_{a}^2} \left( \sum_{j=1}^{m} t_j \right) \left( \sum_{j=1}^{m} t_j \right) + \frac{1}{\sigma_{a}^2} \left( \sum_{j=1}^{m} t_j \right) \left( \sum_{j=1}^{m} t_j \right) \left( \sum_{j=1}^{m} t_j \right)
\]

We have \( m \) observations spread over \( m - 1 \) equal time intervals spanning \( T \). But
\[
\sum_{j=1}^{\infty} = m \\
\sum_{j=1}^{\infty} i = \Delta t \sum_{j=1}^{\infty} (j - 1) = \Delta t \sum_{j=0}^{\infty} j = \frac{T}{m-1} \sum_{j=0}^{\infty} j = \frac{T^2 (m-1)}{2} = \frac{mT}{2} \\
\sum_{j=1}^{\infty} i^2 = \Delta t^2 \sum_{j=1}^{\infty} (j - 1)^2 = (\Delta t)^2 \sum_{j=0}^{\infty} j^2 = (\Delta t)^2 \sum_{j=0}^{\infty} j = \frac{T^2}{(m-1)^2} \frac{(m-1)m(2m-1)}{6} \\
= \frac{T^3 m(2m-1)}{6(m-1)}
\]

so that
\[
\sum_{j=1}^{\infty} A^T W A = \frac{1}{\sigma_a^2} \begin{pmatrix} m & \frac{mT}{2} \\ mT & T^2 m(2m-1) \end{pmatrix} = \frac{m}{\sigma_a^2} \begin{pmatrix} 1 & \frac{T}{2} \\ \frac{T}{2} & T^2 (2m-1) \end{pmatrix}
\]

Similarly, for the radar observations
\[
\sum_{j=1}^{\infty} A^T W A = \frac{k}{\sigma_\phi^2} \begin{pmatrix} 1 & \frac{T}{2} \\ \frac{T}{2} & GH^2 + T^2 (2k-1) \end{pmatrix}
\]

Combining gives
\[
\sum_{j=1}^{k} A^T W A = \begin{pmatrix} m \frac{k}{\sigma_a^2} \frac{\sigma_\phi^2}{\sigma_\phi^2} & \frac{T}{2} \left( \frac{m}{\sigma_a^2} \frac{k}{\sigma_\phi^2} \right) \\ \frac{T}{2} \left( \frac{m}{\sigma_a^2} + \frac{k}{\sigma_\phi^2} \right) & kGH^2 + T^2 \frac{m(2m-1)}{6} + \frac{k(2k-1)}{(k-1)\sigma_\phi^2} \end{pmatrix}
\]

Introduce the notation
\[
F = \frac{m}{\sigma_a^2} + \frac{k}{\sigma_\phi^2} \\
E = \frac{m(2m-1)}{(m-1)\sigma_a^2} + \frac{k(2k-1)}{(k-1)\sigma_\phi^2} = \frac{m}{\sigma_a^2} \left( 2 + \frac{1}{m-1} \right) + \frac{k}{\sigma_\phi^2} \left( 2 + \frac{1}{k-1} \right)
\]

Then
\[
\sum_{j=1}^{k+\mu} A^j W A = F \left( \frac{T}{2} - \frac{k^2 E}{6F} + \frac{T^2}{F^2 \sigma^2_p} \right)
\]

(11)

Eq. (6) can be solved for \( \Delta X \) to give

\[
\Delta X = (A^T W A)^{-1} A^T W B
\]

(12)

At this point we need to compute the inverse of the matrix in Eq. (11). Let

\[
D = F \left( \frac{k^2 E}{6F} + \frac{T^2}{4} \right)
\]

Then the inverse is

\[
(A^T W A)^{-1} = \frac{1}{D} \begin{pmatrix}
\frac{k^2 E}{6F} + \frac{T^2}{4} & -\frac{T}{2} \\
-\frac{T}{2} & 1
\end{pmatrix}
\]

(13)

The least squares correction to the initial orbital elements is given by Eq. (12)

\[
\Delta X = \frac{1}{D} \begin{pmatrix}
\frac{k^2 E}{6F} + \frac{T^2}{4} & -\frac{T}{2} \\
-\frac{T}{2} & 1
\end{pmatrix} \begin{pmatrix}
\frac{T}{m-1} P_r \\
\frac{T}{k-1} Q_r
\end{pmatrix}
\]

which gives

\[
\Delta M_\theta = \frac{1}{D \sigma^2_{\theta}} \left( \frac{k^2 E}{6F} + \frac{T^2}{4} \right) P_r - \frac{T^2}{2(m-1)} P_r \\
+ \frac{1}{D \sigma^2_{\theta}} \left( \frac{k^2 E}{6F} + \frac{T^2}{4} \right) Q_r - \frac{T^2}{2(k-1)} Q_r
\]

\[
\Delta n_\theta = \frac{1}{D \sigma^2_{\theta}} \left( \frac{PT}{2} + \frac{P_r T}{m-1} \right) + \frac{1}{D \sigma^2_{\theta}} \left( \frac{QT}{2} + GHR + \frac{Q_r T}{k-1} \right)
\]
COVARIANCE

The prediction errors can be estimated using the covariance matrix. The covariance matrix at a time $t$ is given by

$$C(t) = A_t (A^T W A)^t A_t^T$$  \hspace{1cm} (14)

where the subscript $t$ indicates evaluation at the desired prediction time and where

$$A_t = \begin{pmatrix} 0 & H \\ 1 & T + t \\ 1 & T + t \end{pmatrix}$$

The covariance is

$$C(t) = \begin{pmatrix} 0 & H \\ 1 & T + t \\ 1 & T + t \end{pmatrix} \frac{1}{D} \begin{pmatrix} kH^2 + \frac{T^3}{6F} + \frac{T}{2} \\ \frac{TH}{F} \frac{T}{2} + 1 \\ H & T + t & T + t \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ \sigma_{pp}^2 & \sigma_{pp} \sigma_{pa} & \sigma_{pa}^2 \\ \sigma_{pa}^2 & \sigma_{pa} \sigma_{aa} & \sigma_{aa}^2 \end{pmatrix}$$

$$C(t) = \begin{pmatrix} \sigma_{pp}^2 & \sigma_{pp} \sigma_{pa} & \sigma_{pa}^2 \\ \sigma_{pp} \sigma_{pa} & \sigma_{pp} \sigma_{aa} & \sigma_{pa} \sigma_{aa} \\ \sigma_{pa} \sigma_{aa} & \sigma_{pa} \sigma_{aa} & \sigma_{aa}^2 \end{pmatrix}$$

where

$$\sigma_{pp}^2 = \frac{H^2}{D}$$

$$\sigma_{pp} \sigma_{pa} = \sigma_{pa} \sigma_{aa} = \sigma_{\alpha\alpha}^2 = \frac{1}{D} \left( Tt + t^2 + \frac{kH^2}{F \sigma_p^2} + \frac{T^3}{6F} \right)$$  \hspace{1cm} (15)

$$\sigma_{pa}^2 = \frac{1}{D} \left( \frac{TH}{2} + tH \right)$$

These terms represent the expected variances and covariances. Let us simplify some terms to provide better insight into their nature.

$$\frac{1}{D} \left( Tt + t^2 + \frac{kH^2}{F \sigma_p^2} + \frac{T^3}{6F} \right) = \frac{1}{D} \left( \frac{kH^2}{\sigma_p^2} + \frac{T^3}{6} - \frac{T^2 F}{4} \right) = \frac{1}{DF} \left( \frac{kH^2}{\sigma_p^2} + \frac{T^3}{6} - \frac{T^2 F}{4} \right)$$
Then Eq. (15) becomes

\[
\frac{1}{D} \left( Tt + t^2 + \frac{kH^2}{F \sigma_p^2} + \frac{T^2 E}{6F} \right) = \frac{Tt}{D} + \frac{1}{F} + \frac{T^2}{4}
\]

\[
\frac{1}{D} \left( Tt + t^2 + \frac{kH^2}{F \sigma_p^2} + \frac{T^2 E}{6F} \right) = \frac{1}{D} \left( \frac{P}{4} + Tt + t^2 \right) + \frac{1}{F}
\]

Then Eq. (15) becomes

\[
\sigma_{\psi_0}^2 = \frac{H^2}{D}
\]

\[
\sigma_{\theta_0}^2 = \sigma_{\alpha x}^2 = \sigma_{\beta x}^2 = \frac{1}{D} \left( \frac{T^2}{4} + Tt + t^2 \right) + \frac{1}{F}
\]

\[
\sigma_{\rho_0}^2 = \sigma_{\rho a}^2 = \frac{1}{D} \left( \frac{TH}{2} + tH \right)
\]

EQUATION SUMMARY

At this point, let us summarize the formulas in one place.

\[
\Delta M = \frac{1}{D \sigma_a^2} \left( \frac{kH^2}{F \sigma_p^2} + \frac{T^2 E}{6F} \right) - \frac{T^2}{2(m-1)} P - \frac{1}{D \sigma_a^2} \left( \frac{kH^2}{F \sigma_p^2} + \frac{T^2 E}{6F} \right) Q - \frac{T^2}{2(k-1)} O
\]

\[
\Delta n_0 = \frac{1}{D \sigma_0^2} \left( \frac{PT}{2} + \frac{P_T T}{m-1} \right) + \frac{1}{D \sigma_0^2} \left( \frac{QT}{2} + GHR + \frac{Q_T T}{k-1} \right)
\]

\[
\sigma_{\psi_0}^2 = \frac{H^2}{D}
\]

\[
\sigma_{\theta_0}^2 = \sigma_{\alpha x}^2 = \sigma_{\beta x}^2 = \frac{1}{D} \left( \frac{T^2}{4} + Tt + t^2 \right) + \frac{1}{F}
\]

\[
\sigma_{\rho_0}^2 = \sigma_{\rho a}^2 = \frac{1}{D} \left( \frac{TH}{2} + tH \right)
\]

\[
E = \frac{m}{\sigma_a^2} \left( 2 + \frac{1}{m-1} \right) + \frac{k}{\sigma_0^2} \left( 2 + \frac{1}{k-1} \right)
\]
Consider a mix of radar and optical tracking. We will rearrange previously derived equations to provide greater insight into the effects of the mixed data. From the summary Eqs. (17)

\[
D = \frac{kH^2}{\sigma_p^2} + \frac{T^2 E}{6} - \frac{T^2 F}{4}
\]

Substitute for \( E \) and \( F \) from Eq. (17) to get

\[
D = \frac{kH^2}{\sigma_p^2} + T^2 \left( \frac{m}{12\sigma_p^2} \frac{m+1}{m-1} + \frac{k}{12\sigma_p^2} \frac{k+1}{k-1} \right)
\]
Now we will examine the coordinate variances.

\[ D = \frac{kH^2}{\sigma^2_\rho} + \frac{k}{\sigma^2_\phi} \left( \frac{k+1}{k-1} \right) \frac{T^2}{12} + \frac{m}{\sigma^2_\omega} \left( \frac{m+1}{m-1} \right) \frac{T^2}{12} \]  

Now we will examine the coordinate variances.

\[ \sigma^2_{\rho_0} = \frac{H^2}{D} \]
\[ \sigma^2_{\phi_0} = \sigma^2_{\omega_0} = \frac{1}{D} \left( \frac{T^2}{4} + Tt + t^2 \right) + \frac{1}{F} \]  

We begin by making some general observations about the behavior of the predicted coordinate variances.

- Coordinate variances are directly proportional to the measurement variances
- Coordinate variances are inversely proportional to the number, \(k+m\), of measurements
- Coordinate variances are inversely proportional to the square of the fit span, \(T\)
- The radial variance is a constant
- The in-track variance grows quadratically with time

Each of these comments shows that the results of Eqs. (19) match very nicely with our intuition of how an orbit determination should work.

Let us assume that the total number of observations \(k+m\) is fixed. We would like to examine the question of how a mix of radar and optical tracking influences the quality of the resulting orbital prediction. The mix of observations and their quality only enter the variances in Eq. (19) through the denominators, \(D\) and \(F\). Making the denominators as large as possible will decrease the variances at epoch as well as for future times. We further note that the terms with denominator \(F\) do not depend on time. So basically, they provide a constant bias to the angle variances. On the other hand the terms with denominator \(D\) will grow quadratically as prediction time increases. Thus, the terms with denominator \(D\) will eventually dominate the behavior of the variances.

Before looking at the dominant \(D\) related terms, let us briefly examine the bias term.

\[ F = \frac{m}{\sigma^2_\alpha} + \frac{k}{\sigma^2_\phi} \]

Generally, the angle quality of an optical tracker is better than the angle quality of a radar tracker (by about a factor of 10). Thus, we can assume
\[ \sigma^2_\rho < \sigma^2_\phi \]

and hence

\[ \frac{1}{\sigma^2_\rho} > \frac{1}{\sigma^2_\phi} \]

Since our desire is to maximize \( F \), we would tend to select all optical observations and no radar observations.

Now consider \( D \). We have previously shown that

\[ D = \frac{kH^2}{\sigma^2_\rho} + \frac{k}{\sigma^2_\phi} \left( \frac{k+1}{k-1} \right) \frac{T^2}{12} + \frac{m}{\sigma^2_\phi} \left( \frac{m+1}{m-1} \right) \frac{T^2}{12} \]

Now, we substitute

\[ H = \frac{2a_s}{3n_o} \]
\[ n_o = \frac{2\pi}{P} \]
\[ T = iP \]

where

\( P = \) satellite period
\( i = \) number of revs in fit span

Then

\[ D = \frac{a_s^2P^2}{3} \left[ k \frac{1}{3\pi^2} \frac{1}{\sigma^2_\rho} + k \left( \frac{k+1}{k-1} \right) \frac{i^2}{4a_s^2\sigma^2_\phi} + m \left( \frac{m+1}{m-1} \right) \frac{i^2}{4a_s^2\sigma^2_\phi} \right] \]

Now our issue is how to distribute the total observations between \( k \) (radar) and \( m \) (optical). Since we will choose \( k \) and \( m \) based on the other variables in the terms, we really need to compare the sizes of the quantities

\[ \frac{1}{3\pi^2} \frac{1}{\sigma^2_\rho} + \frac{i^2}{4a_s^2\sigma^2_\phi} \quad \text{versus} \quad \frac{i^2}{4a_s^2\sigma^2_\phi} \]
Since the issue is how to best use the high accuracy measurements (radar range and optical angle) we will examine a range of values for these measurements. We will vary the radar range quality between 10 and 50 meters. We will vary the optical angle quality between 1 and 20 arc seconds. We will parametrically vary the fit length between 1 day and 20 days. The vertical scale is the radar term minus the optical term for the quantity \( D \). So a positive number means use radar and a negative number means use optical. Since the radar angle is not generally assumed to be of high quality, we will not include it in the parameterization. Rather, we will assume a fixed nominal value of 

\[ \sigma_\theta = 0.01 \text{ degrees} \]

We first examine a semi-synchronous orbit displayed in Figures 2 through 6.

![Radar - Optical Mix - 1 day fit - Semi-Sync](image)

Figure 2: Semi-synchronous one day orbit determination
Figure 3: Semi-synchronous three day orbit determination

Figure 4: Semi-synchronous five day orbit determination
Figure 5: Semi-synchronous ten day orbit determination

Figure 6: Semi-synchronous twenty day orbit determination
We now examine synchronous orbits displayed in Figures 7 through 11.

Figure 7: Synchronous one day orbit determination
Figure 8: Synchronous three day orbit determination

Figure 9: Synchronous five day orbit determination
Figure 10: Synchronous ten day orbit determination

Figure 11: Synchronous twenty day orbit determination
Thus, we have an explicit way to determine the optimum use of radar and optical sensors for orbit determination. As an example, suppose that our sensors have the following accuracy characteristics.

\[
\begin{align*}
\sigma_\rho &= 30 \text{ meters} \\
\sigma_\theta &= 0.01 \text{ degrees} \\
\sigma_\kappa &= 0.0014 \text{ degrees} = 5 \text{ arc sec}
\end{align*}
\]

Then we would conclude the following from our analysis.

<table>
<thead>
<tr>
<th>Orbit type</th>
<th>Fit Span (days)</th>
<th>Best Mix</th>
</tr>
</thead>
<tbody>
<tr>
<td>Semi-synchronous</td>
<td>1</td>
<td>All Radar</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>All Optical</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>All Optical</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>All Optical</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>All Optical</td>
</tr>
<tr>
<td>Synchronous</td>
<td>1</td>
<td>All Radar</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>All Radar</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>All Radar</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>All Optical</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>All Optical</td>
</tr>
</tbody>
</table>

Practically speaking, radar resources are limited for deep space tracking. So a recommendation for “All Radar” means use as much radar resources as are available.

**OBSERVATION NOISE EFFECT ON ORBIT DETERMINATION CORRECTIONS**

From Eq (17) the corrections to the elements that come from the orbit determination are

\[
\Delta M_\kappa = \frac{1}{D \sigma_\kappa^2} \left[ \frac{kH^2}{F \sigma_\rho^2} + \frac{T^2 E}{6F} \right] - \frac{T^2}{2(m-1)} P_r + \frac{1}{D \sigma_\theta^2} \left( \frac{kH^2}{F \sigma_\rho^2} + \frac{T^2 E}{6F} \right) Q - \frac{T G H R}{2} \frac{T^2}{2(k-1)} Q_r
\]

\[
\Delta n_\kappa = \frac{1}{D \sigma_\kappa^2} \left( -\frac{P T}{2} + \frac{P_r T}{m-1} \right) + \frac{1}{D \sigma_\theta^2} \left( \frac{Q T}{2} + G H R + \frac{Q_r T}{k-1} \right)
\]

First of all, recall that we assumed no modeling errors in our solution. Thus, the corrections from the orbit determination should be zero. These corrections result purely
from the measurement noise and appear through the terms $P, Q, R, P_r, Q_r$. Since we have assumed our measurement statistics have zero mean, we would expect that the measurement noise effect on the orbit corrections should be very small and should approach zero as we increase the number of measurements.

The terms $P, Q, R, P_r, Q_r$ depend on the specific errors that occurred at each observation. These terms are of two types. The first three terms

$$P = \sum_{j=1}^{k} \sigma_a,$$

$$Q = \sum_{j=1}^{k} \sigma_\phi,$$

$$R = \sum_{j=1}^{k} \sigma_{\rho_j}$$

are simply the sums of the specific measurement errors. Since the measurement errors have zero mean, one can expect these three summations to be near zero for a large number of observations.

Thus, the orbit determination corrections reduce to

$$\Delta M_\phi = -\frac{1}{2D} \frac{T^3}{(m-1)} \left( \frac{1}{\sigma_a^2} + \frac{1}{\sigma_\phi^2} \right)$$

$$\Delta n_\phi = \frac{T}{D} \left( \frac{1}{m-1} \frac{P_r}{\sigma_a^2} + \frac{1}{k-1} \frac{Q_r}{\sigma_\phi^2} \right)$$

The remaining terms

$$P_r = \sum_{j=1}^{k} (j-1) \sigma_a,$$

$$Q_r = \sum_{j=1}^{k} (j-1) \sigma_\phi,$$

are very interesting. Each measurement error is scaled (through the index $j$) by where in the orbit determination interval that it occurred. In other words, the measurement errors are weighted by time. A large measurement error has more or less effect depending on where it occurred. Several interesting things can be said about these time correlated terms.
• These least squares corrections result purely from measurement noise
• Since we have assumed we started with the truth element set, these “corrections” will cause degraded predictions
• The correction to mean anomaly will just cause a small constant bias in the location prediction
• However, the correction to the mean motion will cause an effect which grows linearly with prediction time
• The time correlation effect causes the measurement errors that occur further from epoch to have a greater impact than those that happen closer to epoch
• This time correlated measurement effect disappears if we use only range measurements

Using the nominal noise statistics from the previous example, we see that errors induced in the mean motion due purely to measurement noise can produce prediction errors of a kilometer or more after 20 days. So these errors can be substantial. This source of error even in the presence of a perfect dynamic model and perfectly normal, zero mean, measurement errors is quite surprising and intriguing.

CONCLUSIONS

For circular satellite motion synchronized with the Earth motion, we are able to formulate the covariance analytically. This formulation is best suited for high level trades in order to better focus extensive numerical investigations. The analytical expressions reveal several interesting properties of the covariance. The formulation allows determination of the optimum choice of resource use between radar and optical measurements. It also reveals that the optimum mix depends not only on sensor quality but also on fit span length. Finally, the analytic formulation unveils a curious and surprising link between the orbit determination corrections and where each measurement error occurs in time.

REFERENCES