HIGH-ORDER STATE FEEDBACK GAIN SENSITIVITY CALCULATIONS USING COMPUTATIONAL DIFFERENTIATION

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A nonlinear feedback control strategy is presented where the feedback control is augmented with feedback gain sensitivity partial derivatives for handling model uncertainties. Derivative enhanced optimal feedback control is shown to be robust to large changes in the model parameters. The OCEA (Object Oriented Coordinate Embedding) computational differentiation toolbox is used for automatically generating first- through fourth-order partial derivatives for the feedback gain differential equation. Both linear and nonlinear scalar applications are presented. The model sensitivities are obtained about a nominal reference state by defining the Riccati differential equation as being derivative enhanced: OCEA then automatically generates the first- through fourth-order Riccati gain gradients. An estimator is assumed to be available for predicting the model parameter changes. The optimal gain is then computed as a Taylor series expansion in the Riccati gain as a function of the system model parameters. The pre-calculation of the sensitivity gains eliminates the need for gain scheduling for handling model parameter changes. Examples are presented that demonstrate the impact of nonlinear response behaviors, as well as the effectiveness of the generalized sensitivity enhanced feedback control strategy.

INTRODUCTION

Feedback control is a powerful methodology for handling model and parameter uncertainty in real-world applications. The calculations required for developing optimal solutions, however, demand significant analyst time and computational resources. Given a useful nominal plant model for developing the control approach, it is well-known that optimal solutions only perform well for a limited range of model and parameter uncertainty before the system response degrades and the resulting control objectives are not met. By augmenting the feedback control solution with sensitivity calculations, the modified control system uses a truncated Taylor series to handle the model and parameter changes. This approach is shown to significantly expand the range of model uncertainty that can be handled while still maintaining desired controlled performance goals. Alternatively, one would have to re-compute optimal feedback gains each time large changes arise in either the model or parameter values, or develop a gain-scheduling strategy. The feedback gains and sensitivity partial derivatives are pre-computed as a one-time calculation. The plant is described by the first order state differential equation \( \dot{x} = f(x, p; t) \), where \( x \in \mathbb{R}^N \) denotes the state vector, and \( p \in \mathbb{R}^M \).
denotes the parameter state vector. It is important to realize that the parameter variations are assumed to be time-varying (i.e. \( p = p(t) \)). Several examples are presented that demonstrate that the proposed sensitivity-enhanced feedback control formulation handles a wide-range of applications that are characterized by time-varying model and parameter uncertainties with a single calculation.

Potential sources of model uncertainty include, but are not limited to: expendable fuels, articulated moving parts (i.e., pointing subsystems, rotating machinery, scanning systems, etc.), time-varying stiffness and damping behaviors, changing moments of inertia, robotic manipulations of external objects, environmental effects, etc. A key step in the control design process is concerned with establishing the expected ranges for the model and parameter uncertainties. Control design iterations establish the number of sensitivity partial derivative terms to be retained in the control system design for handling expected range in model uncertainty. This approach trades computer memory for storing the gain gradients, versus the real-time re-computation of the control gains. By pre-computing sensitivity gains it is anticipated that many real-time applications can be handled that are experiencing large time-varying plant changes. The state estimation for the feedback control must be generalized to include estimates of all of the parameters where sensitivities are computed.

In a classical control approach, given a plant model, one develops an optimal control strategy by defining an integral-based performance index that penalizes both the state and control time histories. The calculus of variations provides the necessary condition for the optimal control problem. One obtains a time-varying Riccati matrix differential equation for the gain matrix calculations, and the control is defined by multiplying the Riccati gain matrix with the state vector. Several authors have attempted generalizations of feedback control where the co-state variable is expanded as a power series in the state variable. Recently, Majji, Turner, and Junkins\(^1\), \(^2\) have explored the developments of this theory to high-order. Malanowski and Maurer\(^3\), \(^4\) introduced a first-order sensitivity method to investigate the parametric variation of solutions to constrained optimal control. Later work considers theoretical issues such as regularity. Pinho and Rosenblueth\(^5\) utilize the implicit function theorem to transform a constrained optimal control problem into an unconstrained form and propose a solution approach. Recent work by McCrate\(^6\) utilized higher-order differentiations of the Hamilton-Jacobi-Bellman (HJB) equation to solve nonlinear optimal control problems and their sensitivities. The success of these earlier approaches motivates our investigation into developing high-order sensitivity models.

In particular, this paper builds on earlier work by Carrington and Junkins\(^7\), \(^8\) and later by Bani Younes et al.\(^9\), \(^10\) Carrington and Junkins expanded the co-state variable as a power series in the state variable. Bani Younes et al. formulated a nonlinear tracking problem, where the co-state variable is expanded as a power series in the tracking error relative to a reference trajectory. A further generalization has been considered where the control strategy has been further enhanced by augmenting the control gains with sensitivity partial derivatives for the system parameters and model errors.\(^11\), \(^12\) This approach, however, proved to be very cumbersome to implement because the gain sensitivity calculations require very complicated array algebra\(^13\), \(^14\) calculations for high-order gains. A major contribution of this paper is the demonstration that the limitations of these earlier efforts are shown to be easily overcome by replacing the gain calculations with a straightforward Taylor series, where computational differentiation tools handle all of the complicated sensitivity calculations in a hidden way, where no array algebra calculations are required. The methodology of this paper is expected to be broadly useful for applications in science and engineering.
Computational Differentiation

Turner’s OCEA\textsuperscript{11,12} Computational Differentiation (CD) software tool builds all of the partial derivative models. OCEA acts as a Language extension for FORTAN 95/2003 when it is linked to application codes. Partial derivative capabilities are enabled by making extensive use of operator-overloading, where multiple levels of the chain rule of calculus are embedded in both the intrinsic (i.e., -, +, /, \* operations) and the standard library functions (i.e., \sin, \cos, \log, \exp, \tan, \cosh, etc. ). Internally, OCEA employs a differential \emph{n-tuple} data structure for storing, manipulating, and computing exact partial derivative models, where a \emph{4-th} order partial derivative differential \emph{n-tuple} data structure is defined by, $f := (f, \nabla f, \nabla^2 f, \nabla^3 f, \nabla^4 f)$. All partial derivative operations are completely hidden from the user. The data structure represents an \emph{abstract compound data object} consisting of a concatenation of $f$ and its first \emph{four} gradient operations. The user only sees the $f$ variable: the gradient operators are referenced individually using the component selector character percent (\%) and the name assigned to the individual gradient operators. Hidden routines carry out all operations for building the gradient operators.

OCEA’s partial derivative calculations are exact because the chain rule of calculus is embedded in the overloaded intrinsic and standard library functions; thereby ensuring that mathematically correct calculations are performed at all derivative orders. The resulting numerical calculations are accurate to the working precision of the machine. At compile time OCEA derives and assembles the partial derivative calculations by invoking the hidden chain rules, the results are saved as an executable program for the calculations. The compiler generates an exact sensitivity solution each time the software is revised. The process is very fast and has the enormous benefit that the all parts of the solution represent the most current form of the solution. Of course, an overhead cost exists for invoking OCEA, however, the run-time overhead penalty more than offsets analyst time saved because partial derivative models do not have to be derived, coded, and validated (frequently saving weeks to months of effort).

\textbf{MATHEMATICAL MODEL AND APPLICATIONS PARAMETER VARIATIONS}

Optimal control methodologies are presented for demonstrating the effectiveness of the proposed sensitivity-enhanced feedback control approach:

\textbf{Sensitivity Calculations for Closed Loop Control}

Consider the scalar problem of minimizing the fixed final time cost function\textsuperscript{1,2}

$$
\min J = \frac{1}{2} s f x^2(t_f) + \frac{1}{2} \int_{t_0}^{t_f} \left( q x^2(\tau) + ru^2(\tau) \right) d\tau
$$

subject to:

$$
\dot{x}(t) = a^0 x(t) + bu(t)
$$

where $x(t_0)$ is given and $t_f$ is fixed. The control solution for this problem is the linear state feedback control $u(t) = -\frac{b}{r} s(a^0, t)x(t)$, where the time varying state feedback gain is calculated by solving the Riccati differential equation backwards in time.

$$
\dot{s} = -2a^0 s + s^2 \frac{b^2}{r} - q
$$
where the final boundary condition is $s(a^0, t_f) = s_f$. Even for this simple problem, it is noteworthy that even a small change in the plant parameter (say $a = a^0 + \delta a$) forces a re-calculation of the state feedback gains for maintaining the desired controlled system response. From Eq. (3) it is obvious that the state-feedback gain is defined by an implicit function of the form $s = s(a, b, r, q)$. Mathematically, parameter sensitivity is handled by developing Taylor series expansions for the state response and control gains about prescribed nominal values, leading to

$$
x(a, t) = x(a^0, t) + \left( \frac{\partial x}{\partial a} \right)_{a=a^0} \delta a + \frac{1}{2!} \left( \frac{\partial^2 x}{\partial a^2} \right)_{a=a^0} (\delta a)^2 + \cdots
$$

$$
s(a, t) = s(a^0, t) + \left( \frac{\partial s}{\partial a} \right)_{a=a^0} \delta a + \frac{1}{2!} \left( \frac{\partial^2 s}{\partial a^2} \right)_{a=a^0} (\delta a)^2 + \cdots
$$

where the partial derivative models for $\frac{\partial x}{\partial a}, \frac{\partial^2 x}{\partial a^2}, \frac{\partial s}{\partial a}, \frac{\partial^2 s}{\partial a^2}$ are obtained by simultaneously numerically integrating Eqs.(3) and (4) along with the following two sets of partial differential equations for the state and control gain sensitivities:

$$
\frac{\partial \tilde{x}}{\partial a} = x + a \frac{\partial x}{\partial a}; \quad \frac{\partial \tilde{x}}{\partial a}\bigg|_{t=t_0} = 0; \quad \frac{\partial^2 \tilde{x}}{\partial a^2} = 2 \frac{\partial x}{\partial a} + a \frac{\partial^2 x}{\partial a^2}; \quad \frac{\partial^2 \tilde{x}}{\partial a^2}\bigg|_{t=t_0} = 0; \quad (5)
$$

and

$$
\frac{\partial \tilde{s}}{\partial a} = -2s + 2 \left(- a^0 + s \frac{b^2}{r} \right) \frac{\partial s}{\partial a}; \quad \frac{\partial \tilde{s}}{\partial a}\bigg|_{t=t_0} = 0
$$

$$
\frac{\partial^2 \tilde{s}}{\partial a^2} = -2 \frac{\partial s}{\partial a} + 2 \left(- 1 + \frac{\partial s}{\partial a} \frac{b^2}{r} \right) \frac{\partial s}{\partial a} + 2 \left(- a^0 + s \frac{b^2}{r} \right) \frac{\partial^2 s}{\partial a^2}; \quad \frac{\partial^2 \tilde{s}}{\partial a^2}\bigg|_{t=t_0} = 0 \quad (6)
$$

These equations are derived analytically, however, OCEA automatically generates numerical solutions for these partial derivatives that can be numerically integrated.

**Vector Case**

The general case of a multivariable systems leads to tensorial equations for the state, control gains, and sensitivity partial derivatives. For simplicity we first consider the case of computing the first-order sensitivity of the state feedback gain matrices with respect to the matrix of plant parameters. Given the state differential equation $\dot{x} = A_0 x + B u$, it is well-known that the stabilizing control is defined $u(t) = -R^{-1} B^T S(t)x(t)$, where the feedback gain matrix $S(t)$ is obtained by numerically integrating the Riccati matrix differential equation: $\dot{S} = -A^T S - SA + SBR^{-1}B^T S - Q$, by assuming the $S, A, B, R, Q$ are derivative enhanced, one simply computes the state equation, where OCEA builds $\dot{S} := \left( \nabla^4 \dot{S}, \nabla^3 \dot{S}, \nabla^2 \dot{S}, \nabla \dot{S}, \dot{S} \right)$. This equation is numerically integrated to provide the desired gain solutions. The implemented feedback control is defined by $u(t) = -R^{-1} B^T \left\{ S + \sum_{n=1}^{4} \frac{1}{n!} \nabla^n S \delta p^n \right\} x(t)$, where $\delta p^n$ denotes an $n$-th order tensor product for the parameter variations. Here we are trading memory storage for computational speed for integrating the matrix Riccati equation.

**Numerical Example-Linear** The linear model of Eq.(2) is simulated by assuming that at $a = b = r = q = 1; \delta a = 0.01$ and $t_f = 5$ seconds, where $x(0) = 1$. The time histories for the state and gains are shown in figures (1a & 1b respectively). You can observe how the enhanced 4-th order gain out performs the classical gain. Figures (1c & 1d) present parameter variations of 50% (a very large change) where the derivative enhanced control easily handles this challenging range in
model uncertainty. Particularly, figure (d) which demonstrates that the terminal error for the fixed time control problem is virtually not impacted by the large parameter variations even though the classical control approach experience quadratic growth in the observed error. This suggests that the new control gain approach is much more robust to modeling errors that classical control designs.

**SENSITIVITY ENHANCED OPTIMAL CLOSED LOOP CONTROL - NONLINEAR SYSTEMS**

The parameter results are now generalized to handle nonlinear terms in the equations of motion. This problem was first considered by Carrington and Junkins\(^7,8\) and later by Bani Younes et al.\(^9\) The nonlinearity is handled by expanding the co-state as a truncated Taylor series in the nonlinear variable, where unknown gain coefficients multiply each of the series expansion terms.

**Scalar Problem**

Consider the optimal control problem of minimizing the following performance index:

\[
\text{Min} J = \frac{1}{2} \int_{t_0}^{t_f} \left\{ qx^2 + ru^2 \right\} \, dt \tag{7}
\]

subject to the nonlinear state equation \(\dot{x} = -(1 + \epsilon)x + \epsilon x^2 + u\). To simplify the technical presentation, the weighting parameters \(q\) and \(r\) are chosen to be 1, though it is fully recognized\(^9,14\) that the
selection of optimal values for \( q \) and \( r \) can improve performance. The Hamiltonian \( H \) is given by

\[
H = \frac{1}{2} \left( x^2 + u^2 \right) + \lambda \dot{x}.
\]

The necessary condition for optimality yields the co-state equations (i.e., \( \dot{\lambda} = -H_x = -x + \left( 1 + \epsilon - 2 \epsilon x \right) \lambda \), and the control \( u = -\lambda \). The system nonlinearity is handled by assuming that the co-state is expressed as following Taylor series

\[
\lambda = k_1 x + k_2 x^2 + k_3 x^3 + k_4 x^4 + \cdots
\]

where the control gains \( k_i \) must be recovered and \( k_2, k_3, \) and \( k_4 \) gains denote new co-state terms for handling the plant model nonlinearities. Introducing Eq. (8) into the Co-state equation and equating powers of \( x^n \) on both sides yields the cascade of gain differential equations

\[
\begin{align*}
\dot{k}_1 &= -1 + k_1^2 + 2 (1 + \epsilon) k_1 \\
\dot{k}_2 &= -3 k_1 \epsilon + 3 (1 + \epsilon + k_1) k_2 \\
\dot{k}_3 &= -4 k_2 \epsilon + 2 k_2^2 + 4 (1 + \epsilon + k_1) k_3 \\
\dot{k}_4 &= -5 k_3 \epsilon + 5 k_2 k_3 + 5 (1 + \epsilon + k_1) k_4
\end{align*}
\]

which account for both the nonlinearity of the state as well as the parameter variations in the plant model. The \( k_1 \) gain equation is the classical Riccati equation. The next section explores how these higher-order gains impact the controlled system response for the nonlinear plant.

**Numerical Example**

The nonlinear model is simulated by assuming that \( \epsilon = 0.01 \) and \( t_f = 5 \) seconds, where \( x(0) = 100 \). Figure (2a) presents the state variable and control histories for different polynomial gain orders. Curves corresponding to higher order are seen to have converged to the optimal control. This conclusion is further confirmed in figure (2b) which indicates that the optimal cost decreases with increasing the co-state polynomial order.

![Figure 2. Enhanced Feedback Solutions for uncertain plant.](image-url)
Gain Sensitivity Calculations  It is obvious that the feedback gain \( k \) is a function of \( \epsilon \). Therefore, any perturbation, \( \delta \epsilon \), will subsequently influence the feedback gain calculations. To handle the gain perturbation induced by the parameter variations we assume the new perturbed plant parameter is given by \( \epsilon = \epsilon^* + \delta \epsilon \), where \( \epsilon^* \) is the nominal value and the numerical values are \( \epsilon^* = 0.01 \) \( \delta \epsilon = \pm 0.009 \). During a control application the extrapolated sensitivity feedback gain is calculated from the truncated Taylor Series expansion:

\[
k_i(\epsilon^* + \delta \epsilon) = k_{i0} + \frac{\partial k_i}{\partial \epsilon} \bigg|_{\epsilon^*} (\epsilon) + \frac{1}{2!} \frac{\partial^2 k_i}{\partial \epsilon^2} \bigg|_{\epsilon^*} (\epsilon)^2 + \frac{1}{3!} \frac{\partial^3 k_i}{\partial \epsilon^3} \bigg|_{\epsilon^*} (\epsilon)^3 + \cdots
\]  

where \( k_i \) denotes the polynomial gain and \( i : 1 \rightarrow 4 \). In the proposed scheme, the feedback gain solution is stored for the nominal gain, as well as the first-through-fourth partial derivatives. Those partials are generated automatically by OCEA, see figure (3a). This allows a large family of nearest neighbor perturbations in the parameter to be handled by a single preliminary calculation. Figure (3b) exhibits the sensitivity in the approximate/calculated solution as a time history. It is obvious that the perturbations in are well-handled at the boundary conditions. Figures (3c & 3d) show that the approximate/calculated final state solution and cost is valid for larger values when compared to the nominal solution.

Figure 3. (a) Feedback Gains. (b) Solution Sensitivity. (c) True Solution when the gain is not updated by \( \delta \epsilon \) vs. approximate/calculated Solution. (d) True Cost when the gain is not updated by \( \delta \epsilon \) vs. approximate/calculated Cost.
SUMMARY AND CONCLUSION

It is well known that the closed loop system dynamics is highly sensitive to plant parameters. Generalizations of classical control results are presented in this paper that mitigates the sensitivity of plant models to both parameter and state nonlinearities. Computational differentiation is introduced as a methodology for automatically generating the required partial derivatives for implementing Taylor series-based generalization of classical feedback control results. Several examples are presented that demonstrate the effectiveness and robustness of the Taylor series-based approach for the feedback gains for nonlinear applications. The derivative enhanced feedback gain solutions are shown to handle large parameter variations with little impact on the resulting terminal errors for the problem. These promising results suggest that the proposed new control strategies can have a significant impact on addressing complex real world applications where parameter uncertainty and model nonlinear effects are present.

ACKNOWLEDGMENT

Any acknowledgments by the author may appear here. The acknowledgments section is optional.

REFERENCES