On the incorrectness of application of the Fourier harmonic analysis for revealing latent periodicities in signals and processes and on a widespread delusion while analyzing optical and radar signatures

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Abstract

Despite our two elucidative (educative) publications on this issue [2, 3], due to a widespread delusion, incorrect applications of the Fourier harmonic analysis for revealing latent periodicities in signals and processes is going on. This problem is of current importance, for instance, while analyzing optical and radar signatures. Here, the problem is investigated in more detail than was in the preceding publications. There were considered demonstrative examples connected with some critical situations, where application of the harmonic analysis with this aim is inadmissible. Some situations and conditions are indicated, where one can use the Fourier expansion for revealing periodicities in signatures with no harmful consequences.

We were impelled to revert to the topic of this presentation by the continuing popular practice of the application of the Fourier harmonic analysis, with the aim of revealing latent periodicities in signals and processes, which is generally incorrect. Especially frequently, one comes across this phenomenon in analysis of biological, medical, economical, astrophysical, and other processes. Particularly, we have it when analyzing photometric and radar signatures, as results of the observation of space objects.

The Fourier’s authority is unquestionable, though the great Frenchman has created his harmonic analysis for absolutely alternative aims. And he evidently did not suspect that it would be applied by his descendants, in cases not answering the purpose.

Earlier, we have already touched the topic, but not so profoundly. Here, we would like

1) in more detail, to reveal the causes of getting incorrect results when using harmonic analysis;
2) to clarify in which cases one can sometimes apply the Fourier expansion for revealing latent periodicities with minimum risk to fail.

It is known from the theory of functions that any continuous, or piece-wise continuous, function given at a finite segment of its argument values, can be fitted with any accuracy in different functional bases, with the help of a polynomial of weighed functions from a given basis. One such basis is a special trigonometric one, namely a finite or infinite set of harmonics with frequencies multiple to some fixed frequency (the lowest one).

The apparatus of Fourier analysis allows presenting any continuous, or piece-wise continuous or generally integrated function, as a sum of such a harmonic series (infinite in a common case), or approximating it by a finite linear polynomial of its harmonics. In this case, as the basic period (the period of the lowest harmonic) one takes the length of the measuring interval, at which the function is given with its known values. All the subsequent periods are the dividers of the basic period.

Remember that in this case, namely an approximation of a function is meant (no extrapolation). Assume that some function was approximated by a segment of the harmonic series with great accuracy. In some research or vital situation, a problem of propagation of a function arises. And in our case, a temptation appears to use the obtained harmonic polynomial as an extrapolative one, with a hope to predict the function behavior outside the approximation interval.

In this position, one should stop and concentrate, and think about the consequences: however ideally the approximation looked, this fact does not imply the same ideal behavior of the polynomial in the extrapolation interval.

For precise prediction of the process in the future, one should first of all recognize and analytically express its organic structure, in other words, to guess a real mechanism of the process origin. In the case of the presence of periodicities in the investigated process, they should be revealed (not approximated by other periodicities, in particular, by terms of the Fourier harmonic series). But what organic essence can be spoken of, if an absolutely alien parameter (the length of a measuring interval) is taken as a basic period? This parameter has no relation to the process structure.

There is something more that can put you on your guard. The high accuracy of an approximation can be reached, not only at the expense of correctly guessing the process analytical structure, but also due to a mechanical increase of the approximation polynomial addends – functions alien to the real physical nature of the process. The same matter is in our case of using the harmonic series for approximation of an arbitrary function. Generally, there may be a random
coincidence of the real period of the lowest periodicity and the length of the measuring interval. Although it is nearly improbable.

The more so, for adequate representation of the process’s physical structure with the help of harmonic series, all the other periods should be dividers of the basic period, which is almost improbable. In real life, different periodicities are stipulated by different causes, usually not related one with another. In the case of a photometric or radar signature the causes of periodicities may be the SO’s attitude dynamics, specific surface geometry, and its mechanical change, if the SO is functional, and so on. So, the real different periods in photometric or radar signals, as a rule, may be even incommensurable.

Because of this fact, such a simple process as a sum of two periodicities, not multiple between their periods, can be well approximated by the harmonic polynomial having more than ten, or even a hundred harmonics, with appreciable amplitudes. And from those, the researcher should select two components with the overwhelming amplitudes – as the sought for solution. The latter is often difficult or impossible. In other words, the process really contains two periodicities but the Fourier transformation suggests several (perhaps tens or hundreds) and none of them is correct.

In such cases, it would be unfair to resent Fourier. In contrast to us, he acted rather consistently. Firstly, he did not mean extrapolation but approximation. His aim was to show that any (at least continuous) function can be accurately presented by an infinite harmonic series, or approximated with any accuracy by a finite segment of this series. And he splendidly solved this problem.

Illegitimate application of the Fourier analysis, for revealing latent periodicities in a process given in a limited interval (which has been formed by random circumstances), resembles the action of a person, who tries to unlock the door with a key by chance found in his pocket.

There are cases when one succeeds to reveal the periodicities in some processes with the help of the Fourier expansion in special circumstances. About this see later.

Let us consider in more detail the causes of appearance of incorrect results in the presentation of some process as a segment of the harmonic series. Let us analyze some typical cases.

1. The process under consideration $X(t)$ contains only one periodic component which period is incommensurable with the length of the measuring interval $L$. In other words, the quantity $L$ is not a multiple of the period $T$, and vice versa if $T > L$. Let, for instance, $L=1$ and $T = 0.75$. For this case, the Fourier basis does not contain any elementary function with a close period. Let the process $X(t)$ consist of
only a pure harmonic and no trend. Then the Fourier expansion of \(X(t)\) looks like this:

\[
\tilde{X}(t) = \sum_{n=1}^{\pi} A_n h_n(\varphi_n, T_n, t) = \sum_{n=1}^{\pi} A_n h_n(\varphi_n, \frac{L}{n}, t)
\]

(1)

where \(h_n\) is a normalized (standard) harmonic, that is a sinusoid with a unit amplitude, phase \(\varphi_n\), period \(T_n\), \(T_1 = L, T_2 = L/2, \ldots, T_n = L/n\), \(A_n\) – the amplitude of \(n\)th harmonic of the expansion, and \(\bar{n}\), in principle, may turn into infinity.

As a matter of fact, when expanding the function \(X(t)\) into the Fourier series one solves an optimization program:

\[
\int_{0}^{L} |X(t) - \tilde{X}(t)| \, dt \rightarrow \min_{\{\varphi_n, A_n\}}
\]

or

\[
\int_{0}^{L} (X(t) - \tilde{X}(t))^2 \, dt \rightarrow \min_{\{\varphi_n, A_n\}}
\]

given some fixed number of harmonics in the Fourier basis. As a fact, to reach the extremum the approximation procedure “manipulates” the parameters \(\varphi_n\) and \(A_n\), id est phases and amplitudes of the harmonics while all \(T_n\) are fixed and “declaratively” determined by the value of \(L\), absolutely not connected with the sought for period \(T\). So, absolutely alien to our task parameter \(L\) manages the search for the best approximation of \(X(t)\).

So, the mechanism of optimization, placed in such rigid frames with highly limited freedom of choice, is compelled to use the functional basis containing only harmonics with the periods equal to dividers of \(L\). The better approximation will be attained due to increasing the segment of the Fourier series. The latter indicates that the poor basis was taken, which did not mathematically reflect the process structure.

This is nearly a paradox: objectively, the process contains only one harmonic, nevertheless a poorly selected method suggests several ones as a solution, none of them concurring with the real sought for one. If the approximation basis consisted from only one harmonic and its period by chance were equal to the sought for period, that would be the best basis, enough for getting the ideal approximation, the more so, for ideal extrapolation as well, which is sometimes more important.
Only in cases when \( L \) is much greater than \( T \) one can satisfactorily fit the function \( X(t) \) at the expense of remote terms of the Fourier series. And only on this condition, the Fourier method applied to this process gives plausible results for this particular process. And of course, when \( L = T \) (which is almost improbable), it gives a correct result.

2. In Figure 1 the simplest sinusoidal process given at the interval containing less than one and a half wave is shown. The spectrogram of the Fourier expansion in Figure 2 shows that the Fourier method could not cope with the task of revealing the one and only harmonic on such a condition. Meanwhile, the special (dedicated) method intended for revealing latent periodicities (\( D \)-method [1, 2]) easily solves the problem [2].

![Fig. 1. Canonical example.](image1)

![Fig. 2. Fourier transformation failure.](image2)

However, if the measuring interval were increased as many as 100 times (which is in practice often inadmissible), the Fourier method would give a correct solution. Though even in this case the \( D \)-method gives a more exact solution [2].

3. Let us take another process which is a sum of two rectangular pulse-like periodical components having periods \( T_1 = 0.8 \text{ s} \) and \( T_2 = 2.4 \text{ s} \), respectively, and equal amplitudes, \( A_1 = A_2 = 10 \). In this example, the measuring interval is equal to \( L = 48 \text{ s} \), which is much greater than the values of both periods (Figure 3).

Nevertheless, even on such a favorable condition, the harmonic analysis gave an absolutely unfit “solution” (see Figure 4): it “revealed” 7 periods instead of two. Two of them are somewhat close to the real ones, but it is impossible to identify them, as their amplitudes are very small compared with the rest. The most powerful harmonic – as if the “main” one (its amplitude being several times more than the others) appears false. All the harmonic composition is incorrect. The
Fourier expansion has failed in this case. Meanwhile, both real components have been revealed with no problem and with absolute precision with the help of the dedicated $D$-method [3].

Figure 3. A sum of 2 rectangular pulse-like periodicities
In this case, the cause of the unfitness of the harmonic analysis for revealing the periodicities is a poor choice (really, the imposing) of a functional basis of approximation, absolutely alien to the process nature.

Meantime, the specialized methods of revealing latent periodicities do not deal with an approximation task at all – no reconstruction of the process. They are intended (and deliberately designed) only for detecting marks of recurrences, or repetitions, in the process being investigated. The form of the sought for periodicity is of no importance. This is the main and principal difference of the specialized (dedicated) methods for revealing latent periodicities from the Fourier expansion of the process segment or any other approximation approaches.

That was a very demonstrative, didactic, and, say, sobering example. As a matter of fact, in real physical processes (and not only physical but economical, biological and others) one usually finds no pure harmonic components. In photometric and radar signatures, due to specular effects, heterogeneity of atmosphere and the SO surface, during rotation and oscillations of the SO, flashes and drops of the signal level occur. As a result, a form of periodicities in signatures significantly differs from a harmonic.
The Fourier analysis has a chance to correctly reveal periodicities only in the next conditions:

1) the measuring interval should be much longer than the longest period of the periodicities in the process;

2) the sought for periodicities should be harmonics.

Otherwise, one cannot count on the correct solution.

Here, the following must be understood. The approximation and propagation problems have different mathematical natures. The externally ideal solution of the first one does not necessarily imply even the mean solution of the second (if the approximation polynomial of the first problem is used as a propagator). The approximation problem can be successfully solved in any sufficiently wide functional basis and any accuracy can be achieved. And the basis used may have nothing to do with the real mechanism of the process origin and with the authentic functional components of its analytical description.

The extrapolation problem, as a second stage of a double problem (as it is usually set using the harmonic analysis for its solution), can be successfully and warrantedly solved, if and only if in the solution of the first stage, the approximation polynomial exactly describes the mechanism having generated the analyzed process. The Fourier transformation does not have such an aim. The propagation program can be used for checking the correctness of mathematically representing the organic structure of the analyzed process with the help of the approximation polynomial.

By the way, in practical situations, the propagation problem may not be addressed (as well as the approximation one). As often happens, the more important is the revelation of the real unknown periods of the process periodical components, for example, to reveal the source, or sources, of the corresponding oscillations encoded in the process signature. In such a manner, the dedicated methods (periodogamanalytic, correlation and others) are constructed. They just give an answer to the question of the composition of the process. The latter is simpler and wiser than to solve the approximation problem, and then the extrapolation one, because for our aim it is unnecessary to exactly reproduce the “exotically” looking periodicities of the process with the help of the approximation polynomial. The “exotic” components may be replaced by simple harmonics. It is enough for revealing the periods of the latent periodicities.

If it arises, the propagation problem is a different problem. In such a case, it is important not only to determine the periods, phases, and amplitudes of the periodicities, but their forms as well, for a precise prediction of the process at every point of the extrapolation interval. Otherwise, the propagated approximation polynomial outside the measuring interval and the real process there, would concur
only in the nodes of the periodicity (if it is only one). Though, as it will be seen later, there can be found a way out of this situation.

Figure 5. An “exotic” process and its conditional approximation and extrapolation

In Figure 5, for the sake of simplicity and obviousness, the process is a periodicity, very differing by its form from a harmonic, and has no trend. The approximation polynomial contains only one harmonic with the rightly guessed period and phase. In this example, one can see that the warranted coincidence of the prediction and the analyzed process occurs only in its nodes (both within the approximation interval and the extrapolation one). At the same time, it is easy to see how to correct the prediction curve to get any accuracy of the prognosis, not only in the nodes but in the extrema as well, and generally at every point of the extrapolation interval.

Let us repeat once more: to get a good result in this case the approximation polynomial should consist of harmonics (sinusoids) having periods equal to those of the real periodicities of the process, but not functions of the Fourier harmonic series. For this aim (id est for correct revelation of the real organic periods of the process components) special dedicated methods exist – the periodogramanalytic, correlation methods, and those of some other categories [4, 5, 1]. And it is not worth-while to take the turned up to your hand Fourier harmonic analysis, elaborated by its author for other tasks different from ours.
References

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