Stability of a chain of phase oscillators

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We study a chain of $N + 1$ phase oscillators with asymmetric but uniform coupling. This type of chain possesses $2^N$ ways to synchronize in so-called traveling wave states, i.e., states where the phases of the single oscillators are in relative equilibrium. We show that the number of unstable dimensions of a traveling wave equals the number of oscillators with relative phase close to $\pi$. This implies that only the relative equilibrium corresponding to approximate in-phase synchronization is locally stable. Despite the presence of a Lyapunov-type functional, periodic or chaotic phase slipping occurs. For chains of lengths 3 and 4 we locate the region in parameter space where rotations (corresponding to phase slipping) are present.

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I. INTRODUCTION

Investigations of synchronized behavior of coupled nonlinear oscillators permeate physics [1,2], chemistry [3,4], biology [5], and engineering [6]. Dating back to Huyghens, this problem has been explored both mathematically and experimentally. The ground-breaking work of Kuramoto [7] led to the realization that synchronization is a ubiquitous behavior in nature and can be explained by simple models for interaction between components of a system (see, for example, the excellent review by Strogatz [8]).

Cohen et al. [9] studied a chain of Kuramoto oscillators to explain the so-called fictive swimming observed in the central pattern generator (CPG) of the primitive vertebrate lamprey. Their purpose was to explain the uniform phase lag along the segmental oscillators. Later, Kopell and Ermentrout [10] proposed a more realistic model for fictive swimming and Williams et al. [11] contrasted this model with experimental observations. Studies of the CPG led to biologically inspired applications in robotics, for example, the autonomous mobile robotic worm of Conradt and Varshavskaya [12] and a turtlelike underwater vehicle of Seo et al. [13].

Cohen et al. [9] observed that in a chain of oscillators the oscillators at the end points play a special role: adaptation of the natural frequency of the end points controls the phase shift between neighboring oscillators throughout the entire chain in the phase-locked equilibrium. (All equilibria of the one-dimensional homogeneous Kuramoto model with periodic boundary conditions were found in [14].)

In this paper we vary this detuning $\delta$ of the end points and the coupling strength (more precisely the ratio $k$ between coupling strengths down the chain and up the chain) to study transitions between phase-locked solutions and phase slipping. If one introduces the phase differences between the oscillators as the new dependent variables, then the phase-locked solutions are equilibria, and for a chain of length $N + 1$ there exist $2^N$ of these equilibria. A transition from one phase-locked solution to another then corresponds to a motion along a heteroclinic connection between the corresponding saddles in the phase space. The $2^N$ equilibria can be classified by an index quantity $\nu_\pi$ which counts how many of the phase differences are equal to $\pi - \delta$. This index turns out to be identical to the number $\nu_\pi$ of unstable dimensions of the saddle (provided $k > -1$) such that connections between equilibria of decreasing index $\nu_\pi$ are generic. Solutions with continuously slipping phases between oscillators show up as rotating waves in the phase space. The periodic phase slipping and its bifurcations can be computed directly if one assumes, for example, pure phase coupling [coupling of the type $K \sin(\theta_j - \theta_{j-1})$ for neighboring oscillators $\theta_j$ and $\theta_{j-1}$]. One of the codimension-one boundaries of rotating waves is a nongeneric connection between equilibria of identical index $\nu_\pi$. We present numerical evidence of this for the cases of $N = 2$ and 3 (that is, the cases of three and four oscillators, respectively). A noticeable difference between these cases is that the basin of attraction for rotating waves appears to be a slightly smaller fraction of the phase space for larger $N$, and the parameter region in the $(k, \delta)$ plane permitting rotations that is, continuous phase slipping) is smaller for the larger $N$. We conjecture that making an oscillator chain longer does not make its propensity for phase slipping larger even if the coupling strength between neighboring oscillators is not increased.

II. MODEL DESCRIPTION

We consider a chain of $N + 1$ phase oscillators with nearest-neighbor coupling as shown in Fig. 1. This model, first proposed by Cohen et al. [9] to explain the fictive swimming observed in the lamprey spinal cord, is given by

$$\frac{d}{dt}\theta_0 = \omega_0 + K_u \Gamma(\theta_1 - \theta_0),$$
$$\frac{d}{dt}\theta_j = \omega_j + K_d \Gamma(\theta_{j-1} - \theta_j) + K_u \Gamma(\theta_{j+1} - \theta_j)$$
for $j = 1, 2, \ldots, N - 1,$
$$\frac{d}{dt}\theta_N = \omega_N + K_d \Gamma(\theta_{N-1} - \theta_N),$$

where $\theta_j$ and $\omega_j = 0$ are the phase and natural frequency of the $j$th oscillator, respectively. The coupling gains are denoted by $K_u$ and $K_d$. The prototypical example for the coupling

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The fixed points of Eq. (4) are given by \( \Gamma(x) = -C^{-1} \Omega \) (\( C \) is invertible for \( k \neq -1 \); see Proposition 2) and correspond to the phase-locked solutions satisfying \( \theta_i - \theta_j = \text{const} \) of the original system (1).

If all components of \(-C^{-1} \Omega\) are less than 1 in absolute value, system (4) has \( 2^N \) equilibria, because of the even symmetry about \( \pi/2 \) \( \Gamma(\pi/2 - x) = \Gamma(\pi/2 + x) \); see Sec. II] of the coupling function. A uniform traveling wave solution (a phase-locked solution with identical phase differences, i.e., \( x = [\delta, \ldots, \delta]^T \)) exists only if the following conditions on the rescaled frequency differences \( \Omega \) are satisfied:

\[
\Omega_i = k \Gamma(\delta), \quad \Omega_j = 0, \quad i = 2, \ldots, N - 1, \quad \Omega_N = \Gamma(\delta).
\]

This means that the frequencies \( \omega_j \) of the original system (1) must be of the form

\[
\omega_j = \omega + K_d \Gamma(\delta), \quad \omega_j = \omega, \quad j = 1, \ldots, N - 1, \quad \omega_N = \omega - K_d \Gamma(\delta),
\]

where \( \omega \in \mathbb{R} \). In other words, all oscillators must have identical natural frequencies except for the two oscillators at the boundary. The “detunings” (differences from the uniform frequency \( \omega \)) of the first and last oscillators are related to one another via the coupling strengths \( K_d \) and \( K_u \).

The two primary parameters affecting the dynamics are the coupling strength ratio \( k \) and \( \delta \). Without loss of generality we can restrict our considerations to the following parameter set:

1. \( \delta \in [0, \pi/2] \): only \( \Gamma(\delta) \) enters the equation and for negative \( \delta \) we can apply the transformation \( x \mapsto -x \) since \( \Gamma \) is odd.
2. \( k \geq -1 \): for \( k < -1 \) we can apply the transformation \( [x_1, \ldots, x_N]_{\text{new}} = [x_N, \ldots, x_1]_{\text{old}} \), \( t_{\text{new}} = k_{\text{old}} t \), and \( k_{\text{new}} = -1/k_{\text{old}} \) such that \( k_{\text{new}} > 0 \). Note that this transformation involves reversal of the time direction (since \( k_{\text{old}} \) is negative) so statements on stability will have to be replaced by the corresponding statements on instability, and vice versa.

We observe that the coupling strength does not enter Eq. (3), which determines the dynamics, at all. Only the ratio between down-chain and up-chain coupling strengths matters. The absolute value of the coupling strength then determines the time scale with respect to the the original time (Eq. (3) is with respect to a rescaled time \( K_u t \)).

IV. STABILITY OF TRAVELING WAVES

Naturally, we are interested in characterizing the local stability of traveling waves, that is, of equilibria of (4). We note that even though the uniform traveling waves can occur only in a slightly degenerate parameter setting, these waves are robust in the sense that slightly nonuniform traveling waves will exist for slight perturbations of these parameter values. For example, nonuniform traveling waves exist for a linear gradient frequency distribution (\( \Omega = \text{const} \)). Cohen et al. [9] derived a necessary condition for the existence of the phase-locked solutions. Ermentrout and Kopell [15] showed the existence of frequency plateaus when this necessary condition is violated.

The linearization of (4) about an equilibrium \( x_o \) is \( J = C \Gamma' (x_o) \), where \( C \) is the coupling matrix given in (5) and

\[
\Gamma'(x_0) = \rho \text{diag}(\sigma_1, \ldots, \sigma_n)
\]

is a diagonal matrix with \( \rho = \Gamma'(\delta) > 0 \) (since \( \delta \in [0, \pi/2] \)) and \( \sigma_i = \pm 1 \), depending on whether the \( i \)th component of \( x_o \) is equal to \( \delta \) or \( \pi - \delta \). Even though the eigenvalues of the matrix \( C \) are known, i.e.,

\[
\lambda_j = -(1 + k) + 2 \sqrt{k} \cos \frac{j \pi}{N + 1}, \quad j = 1, \ldots, N,
\]

\[
\lambda_i = -(1 + k) + 2 \sqrt{k} \sin \frac{j \pi}{N + 1}, \quad j = 1, \ldots, N,
\]

where \( \rho \) is the coupling strength and \( \sigma_i \) is the coupling strength ratio. The eigenvalues of the matrix \( C \) are given by

\[
\lambda_i = -(1 + k) + 2 \sqrt{k} \sin \frac{j \pi}{N + 1}, \quad j = 1, \ldots, N.
\]

where \( \rho \) is the coupling strength and \( \sigma_i \) is the coupling strength ratio. The eigenvalues of the matrix \( C \) are given by
the eigenvalues of the product $C\Gamma'(x_\ast)$ are not known analytically. Therefore, the goal of this section is to establish a simple connection between the number of stable and unstable directions of an equilibrium $x_\ast$ (and thus its stability) of (4) and the number of components of $x_\ast$ equal to $\delta$ and $\pi - \delta$. To establish the result, we associate four indices with each equilibrium $x_\ast$.

\begin{equation}
\begin{aligned}
v_0(x_\ast) &= \text{number of components of } x_\ast \text{ equal to } \delta, \\
v_\pi(x_\ast) &= \text{number of components of } x_\ast \text{ equal to } \pi - \delta, \\
v_u(x_\ast) &= \text{dimension of unstable manifold of } x_\ast, \\
v_s(x_\ast) &= \text{dimension of stable manifold of } x_\ast.
\end{aligned}
\end{equation}

The relation between these indices is given by the following theorem.

**Theorem 1 (dimension of invariant subspaces).** Let $x_\ast$ be an equilibrium of (4). If $k > -1$ then

\begin{equation}
\begin{aligned}
v_0(x_\ast) &= v_\pi(x_\ast) \quad \text{and} \quad v_\pi(x_\ast) = v_u(x_\ast).
\end{aligned}
\end{equation}

**Proof.** Relation (11) is proven indirectly via the Lyapunov-type functional

\begin{equation}
E(x) = \sum_{i=1}^N \left[ \int_0^{x_i} \Gamma(y) y - \Gamma(\delta) x_i \right].
\end{equation}

We observe that the time derivative of $E$ along trajectories of (4) is

\begin{equation}
\begin{aligned}
\dot{E} &= \sum_{i=1}^N [\Gamma(x_i) - \Gamma(\delta)] \dot{x}_i \\
&= -\frac{k+1}{2} \left( [\Gamma(x_1) - \Gamma(\delta)]^2 + [\Gamma(x_N) - \Gamma(\delta)]^2 \\
&\quad + \sum_{i=1}^{N-1} [\Gamma(x_i) - \Gamma(x_{i+1})]^2 \right).
\end{aligned}
\end{equation}

If $k > -1$ then $E(x(t))$ is constant along trajectories; however, it is strictly decreasing for $k > -1$ provided $x(t)$ is not an equilibrium of (4) [i.e., whenever at least one of the phase differences $x_i$ does not satisfy $\Gamma(x_i) = \Gamma(\delta)$]. With this functional $E$ and a bound on its rate of decrease (13) we establish in the Appendix that the linearization $J = C\Gamma'(x_\ast)$ is hyperbolic for $k > -1$. This implies that eigenvalues of $J$ cannot cross the imaginary axis when $k$ is varied. Thus, for all $k > -1$ the Jacobian $J$ is an upper diagonal matrix with its eigenvalues on its diagonal, the number of negative diagonal entries of $J$ is $v_0(x_\ast)$ (the diagonal entry of $C$ is $-1$), and the number of positive diagonal entries of $J$ is $v_u(x_\ast)$.

We observe that for $\delta = 0$ the functional $E(x)$ is bounded, leading for $k \neq 0$ to the result that full synchronization [the equilibrium $(0, \ldots, 0)$ for $\delta = 0$] is globally stable in the same way as the classical results [16].

**V. ROTATING WAVES AND SADDLE CONNECTIONS**

Despite the existence of a Lyapunov functional, the dynamics of (4) is not necessarily trivial because the phase space is an $N$-dimensional torus. Generically we can expect that the unstable manifold of an equilibrium $x$ and the stable manifold of an equilibrium $y$ intersect, giving rise to heteroclinic saddle connections, if $v_u(x) + v_u(y) > N$. This implies that for any equilibrium $x$ heteroclinic connections to all equilibria $y$ satisfying $v_u(y) < v_u(x)$ are generic.

Saddle connections between equilibria of the same type, that is, $v_u(x) = v_u(y)$, are of codimension 1 and can be achieved by tuning the parameters $\delta$ and $k$. These nongeneric saddle connections also form codimension-1 boundaries of periodic orbits of rotating wave type.

Figure 2(a) shows for $N = 2$, $\Gamma(x) = \sin x$, and $\delta = 1$ the dynamics can become chaotic. Numerical evidence for this is shown in Fig. 4 for $N = 3$. Figure 4(a) shows a family of periodic orbits of rotating wave type, $(x_1(T), x_2(T), x_3(T)) = (x_1(0), x_2(0) + 2\pi, x_3(0) + 2\pi)$ where $T$ is the period, $\delta$ is varied, and $k = -0.5$ is kept...
fixed. We observe that this family undergoes a period-doubling sequence. Moreover, the lower (predominantly unstable) part of the branch reaches a homoclinic connection to the saddle \( x_s = (\pi - \delta, \delta, \delta) \) at \( H_s \). The eigenvalues at this saddle have the form \((\mu_+, \mu_+ \pm i \omega_-)\) where \( \mu_+ > 0 \) and \( \omega_- > 0 \) are real numbers and \( \mu_- < \mu_+ < 2\mu_- \). Shil’nikov’s results imply that there is an infinite number of period-doubling cascades of stable periodic orbits close to the homoclinic connection \( H_s \) (under certain nondegeneracy conditions; see [19,20]). The precise sequence of period-\( n \) branches for a homoclinic connection to a saddle of this type and how to calculate them numerically can be found in [21]. The other end of the family of periodic orbits is also a homoclinic connection, to the saddle \((\delta, \pi - \delta, \delta)\), which has three real eigenvalues \( \mu_+ > 0 > -\mu_{-1} > -\mu_{-2} \) where \( \mu_+ > \mu_{-1} \). This implies that there is only a single periodic orbit close to this homoclinic connection (no snaking), and that this periodic orbit is unstable [20].

In the two-parameter plane [shown in Fig. 4(b)] we observe a shape that looks superficially similar to the case \( N = 2 \) except that the range of \( \delta \) over which the region of rotating waves extend is smaller. However, as one can see from Fig. 4(a) the rotating waves are not stable inside the region bounded by the homoclinic connection \( H_s \) and the saddle node. The period-doubling sequence [also shown in Fig. 4(a)] gives a better estimate for the region of stable periodic rotating waves. The remainder of the region is not filled with chaotic rotations because the chaotic attractor at the end of the primary period-doubling cascade collides with heteroclinic saddle connections. Figure 5 shows a periodic motion of period 8, which is close to the end of the period-doubling sequence in parameter space.

Special points in the bifurcation diagram Fig. 4(b) are on the symmetry line \( \delta = \pi/2 \) (where all equilibria collapse to a single degenerate equilibrium) and various degeneracies of the homoclinic connection, for example, the heteroclinic connection \((\pi - \delta, \delta, \delta) \rightarrow (2\pi + \delta, \pi - \delta, \delta) \rightarrow (\pi - \delta, 2\pi + \delta, 2\pi + \delta)\) indicated in Fig. 4(b) (this degeneracy ends the numerical continuation of the homoclinic).

In the original variables \( \theta_j \), the rotating waves shown in this section correspond to a continually drifting phase difference between the oscillators 2 and 3 (for \( N = 2 \)) while the phase difference between oscillators 1 and 2 remains bounded. The homoclinic boundary corresponds to regimes where two neighboring oscillators hover in antiphase for a long time before a phase slip occurs. For \( N = 4 \) the chaotic regimes near the Shil’nikov saddle correspond to the regime

FIG. 3. Bifurcation diagram in the \((k,\delta)\) plane for \( N = 2 \) and \( \Gamma(x) = \sin x \) [computed with (R)AUTO [17,18]].

FIG. 4. Bifurcation diagrams for \( N = 3 \) and \( \Gamma(x) = \sin x \) [computed with (R)AUTO [17,18]]. (a) Family of periodic orbits for fixed \( k = -0.5 \) and varying \( \delta \) and its codimension-one bifurcations. (b) Region with rotating waves in the \((k,\delta)\) plane with its codimension-one bifurcations.

FIG. 5. Rotating periodic orbit of period 8 for \( N = 3 \), \( \Gamma(x) = \sin x , k = -0.5 \), and \( \delta \approx 1.19 \), near the end of the period-doubling sequence in parameter space.
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APPENDIX: PROOF OF HYPERBOLICITY OF EQUILIBRIA

Proposition 2 (absence of eigenvalues with zero real part). Let \( k > -1 \), and let \( x_\ast \) be an equilibrium of (4). Then the Jacobian \( J = C \Gamma'(x_\ast) \) of (4) in \( x_\ast \) cannot have eigenvalues on the imaginary axis.

Proof. First we note that the matrix \( J \) is regular for \( k > -1 \) (and, thus, cannot have an eigenvalue 0):

\[
\det C = \frac{(-1)^N(k^{N+1} - 1)}{k - 1} \neq 0,
\]

\[
\det \Gamma'(x_\ast) = \Gamma'(\delta)^N(-1)^n \neq 0.
\]

We show the absence of purely imaginary eigenvalues indirectly. In preparation for this part of the proof we establish a bound on the gradient of the Lyapunov functional \( E \), given in (12). In an equilibrium \( x_\ast \), the gradient of \( E \) vanishes and the Hessian \( H_\ast \) of \( E \) in \( x_\ast \) is equal to \( \Gamma'(x_\ast) \):

\[
H_\ast = H(x_\ast) = \frac{\partial^2}{\partial x^2} E(x)|_{x=x_\ast}
= \rho \text{diag} (\sigma_1, \ldots, \sigma_n) = \Gamma'(x_\ast).
\]

Consequently, the equilibrium \( (\delta, \ldots, \delta) \) is a local minimum of \( E \), and \((\pi - \delta, \ldots, \pi - \delta)\) is a local maximum of \( E \). Equilibria which have some components equal to \( \delta \) and others equal to \( \pi - \delta \) are saddle points of the graph of \( E \). As we want to study the local stability of an equilibrium \( x_\ast \), we introduce quantities measuring the deviation from \( x_\ast \):

\[
y = x - x_\ast \quad \text{and} \quad D(y) = E(y + x_\ast) - E(x_\ast),
\]

such that \( D(0) = 0 \) and

\[
D(y) = y^T H_\ast y + O(\|y\|^3),
\]

for all \( y \) in a neighborhood of 0. Furthermore, Eq. (13) estimates \( D \) from above for \( k > -1 \):

\[
D(y) \leq -c_0 \|y\|^2,
\]

where \( c_0 > 0 \) is a constant independent of \( y \).

Assume that \( J \) has a purely imaginary eigenvalue, say, \( \lambda = i\mu \) where \( \mu > 0 \), and let \( u \) be the corresponding eigenvector. We choose the time \( t = 2\pi/\mu \). Let \( \epsilon > 0 \)
be sufficiently small (\(\epsilon\) will depend on \(T\)). After time \(T\) the solution of (4) starting from \(x(0) = x_* + \epsilon u\) [let us call the solution \(x(\cdot)\)] satisfies \(x(T) = x_* + \epsilon u + O(\epsilon^2)\). Consequently,

\[
D(x(T)) - D(x(0)) = [x(T) - x_*]^T H_* [x(T) - x_*] + O(\|x(T) - x_*\|^3)
= \epsilon^2 u^T H_* u + O(\epsilon^3) \quad (A6)
\]

On the other hand, the trajectory \(x(t)\) for \(t \in [0,T]\) is a perturbation of order \(\epsilon^2\) of an ellipse with a minimal radius of order \(\epsilon\) around \(x_*\). This means that we can choose a uniform constant \(c_1\) of order 1 such that \(c_1 \epsilon\) is smaller than this minimal radius (\(c_1\) is uniform in \(\epsilon\)) and, hence,

\[
\|x(t) - x_*\| \geq c_1 \epsilon \quad (A7)
\]

for all \(t \in [0,T]\). Inequality (A5) implies that

\[
D(x(T)) - D(x(0)) = \int_0^T \dot{D}(x(t)) dt \\
\leq -c_0 \int_0^T \|x(t) - x_*\|^2 dt \quad (due\ to\ (A5)),
\leq -c_0 T c_1^2 \epsilon^2 \quad (due\ to\ (A7)),
\]

which contradicts (A6) because the constants \(c_0\) and \(c_1\) are independent of \(\epsilon\). ■