A MULTISET-VALUED FIBONACCI-TYPE SEQUENCE

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Abstract
We investigate the structure of a multiset-valued Fibonacci-type sequence created by set union/sum/difference operations. A closed-form generating function is derived to explicitly characterize the elements of these sets and their exact averages and variances. The geometric mean of the sets is numerically demonstrated to exponentially grow, and we prove that the exponent is the square root of the golden ratio.

1. Introduction

One of the simplest examples of random matrix products is the so-called random Fibonacci series (Viswanath [10]). This sequence is generated by the random second-order difference equation

$$x_n = x_{n-1} \pm x_{n-2}, \quad x_0 = x_1 = 1,$$

where the nth term is either the sum or difference of the previous two terms (+ or − are picked independently with probability 1/2 at each step). The structure and growth properties of the random Fibonacci sequence have been studied by several authors (Embree and Trefethen [5]; Sire and Krapivsky [9]; Chan [3, 4]; Janvresse et al. [6]; Kalmár-Nagy [7]; Makover and McGowan [8]; Bai [1]). Viswanath [10] utilized a powerful
combination of random matrix theory and interval arithmetic to show that the sequence almost surely diverges with an exponent of 1.132, i.e., $\nu = \lim_{n \to \infty} \left| \frac{1}{n} \right| x_n \approx 1.132$. Embree and Trefethen [5] numerically investigated $x_{n+1} = x_n \pm \beta x_{n-1}$, a generalization of the random Fibonacci sequence.

The measure of divergence for the random Fibonacci series can be expressed as a path average by considering all the possible values $x_n$ can attain. Let $h_n$ denote the multiset of possible - not necessarily different - values of $x_n$. A multiset is a collection of objects in which elements may occur more than once (Blizard [2]). The exponent (Viswanath’s number) $\nu$ can then be formally expressed as the limit

$$\nu = \lim_{n \to \infty} \left( \prod_{0 \neq x \in h_n} |x|^{1/n} \right)^{1/n} = \lim_{n \to \infty} \left( \prod_{0 \neq x \in h_n} x \right)^{1/(nm)}, \quad (2)$$

where the average (geometric mean) is taken over all the $m$ possible values of $x$ at step $n$ (here $m = 2^{n-2}$).

Even though there is strong correlation between subsequent elements in any particular realization of this random sequence, the same exponent $\nu$ would characterize the random sequence whose $n$th element is randomly chosen from $h_n$. This observation provides the motivation for this study. While a recursion relation for the generating function of $h_n$ can be constructed, the goal of this paper is to provide an even simpler example of a multiset-valued recurrence and to derive some of its basic properties.

2. A Multiset-valued Recurrence

Motivated by studies on the growth of random Fibonacci sequences, here we introduce a multiset-valued Fibonacci-type recurrence. Starting with the two sets $f_1 = f_2 = \{1\}$ the set $f_3$ is constructed as the union of the Minkowski sum and difference of the previous two sets

$$f_3 = (f_2 \oplus f_1) \cup (f_2 \ominus f_1) = \{0, 2\}, \quad (3)$$
where $\oplus$, $\ominus$ denote the Minkowski sum and difference of two sets $X$ and $Y$ defined as

$$X \oplus Y = \{ x + y | x \in X \text{ and } y \in Y \},$$

(4)

$$X \ominus Y = \{ x - y | x \in X \text{ and } y \in Y \} = X \oplus (-Y).$$

(5)

In general, we define

$$f_n = (f_{n-1} \oplus f_{n-2}) \cup (f_{n-1} \ominus f_{n-2}).$$

(6)

Notice that $f_n$ is a multiset, i.e., elements may occur more than once. The number of times an element occurs in a multiset is called its multiplicity (Blizard [2]). To avoid confusion, instead of the standard notation for multisets, here $(\ast n)$ denotes the number of occurrences of an element.

The first few $f_n$’s are given by

$$f_1 = f_2 = \{ 1 \},$$

$$f_3 = \{ 0, 2 \},$$

(7)

$$f_4 = \{ -1, 1, 1, 3 \} = \{ -1, 1(\ast 2), 3 \},$$

$$f_5 = \{ -3, -1(\ast 4), 1(\ast 6), 3(\ast 4), 5 \},$$

$$f_6 = \{ -6, -4(\ast 7), -2(\ast 21), 0(\ast 35), 2(\ast 35), 4(\ast 21), 6(\ast 7), 8 \}. $$

(8)

Few natural questions arise:

What exactly are the elements of $f_n$?

How many elements are there in $f_n$?

What are various statistical properties (mean, variance, geometric mean) of $f_n$?

It is the goal of the next section to answer these questions.

**3. Generating Function and Statistical Properties of $f_n$**

First, we characterize the extremal elements (and thus the range) of $f_n$. In the following, $F_n$ denotes the $n$-th Fibonacci number.
Theorem 1. \( \max f_n = F_n \) and \( \min f_n = 2 - F_n \).

Proof. The first statement follows easily from \( \max f_n = \max f_{n-1} + \max f_{n-2} \) and \( \max f_1 = \max f_2 = 1 \). Then \( \min f_n = \min f_{n-1} - \max f_{n-2} \) = \( \min f_{n-1} - F_{n-2} \). The proposed solution \( \min f_n = 2 - F_n \) satisfies this equation (because of the identity \( F_n = F_{n-1} + F_{n-2} \)), as well as the initial conditions \( \min f_1 = \min f_2 = 1 \).

The next theorem provides a simple expression for the cardinality of \( f_n \).

Theorem 2. \( |f_n| = 2^{F_n-1} \).

Proof. The cardinality of \( f_n \) is expressed as \( |f_n| = |(f_{n-1} \oplus f_{n-2}) \cup (f_{n-1} \ominus f_{n-2})| = |f_{n-1} \oplus f_{n-2}| + |f_{n-1} \ominus f_{n-2}| = 2|f_{n-1}| |f_{n-2}| \). Clearly \( |f_n| = 2^{F_n-1} \) satisfies this recurrence and the initial conditions \( |f_1| = |f_2| = 1 \).

To compute the frequencies of the various integers appearing in \( f_n \), we use the generating function approach (see the excellent treatise by Wilf [12]). In our problem a formal power series

\[
g(n, x) = \sum_{i=-\infty}^{\infty} a_{n,i} x^i
\]

will encode information about \( f_n \). In particular, the index \( i \) will correspond to the integer \( i \) in the set \( f_n \) and the coefficient \( a_{n,i} \) specifies the frequency (number of occurrences) of this integer in \( f_n \). This provides a one-to-one mapping between the set \( f_n \) and the generating function \( g(n, x) \). For example:

\[
f_4 = \{-1, 1(*2), 3\} \leftrightarrow g(4, x) = x^{-1} + 2x + x^3,
\]

\[
f_5 = \{-3, -1(*4), 1(*6), 3(*4), 5\} \leftrightarrow
\]

\[
g(5, x) = x^{-3} + 4x^{-1} + 6x + 4x^3 + x^5.
\]

If the generating function of \( f_n \) is \( g(n, x) \), that of \( -f_n \) is obviously
The union and Minkowski sum/difference of two sets can be easily translated to operations between their generating functions as

\[ f_i \cup f_j \leftrightarrow g(i, x) + g(j, x), \tag{11} \]

\[ f_i \oplus f_j \leftrightarrow g(i, x)g(j, x), \tag{12} \]

\[ f_i \ominus f_j = f_i \oplus (-f_j) \leftrightarrow g(i, x)g\left(j, \frac{1}{x}\right). \tag{13} \]

We are now in the position to express the generating function of \( f_n \) in terms of those for \( f_{n-1} \) and \( f_{n-2} \). Recall that \( f_n = (f_{n-1} \oplus f_{n-2}) \cup (f_{n-1} \ominus f_{n-2}) \) and thus

\[
g(n, x) = g(n - 1, x)g(n - 2, x) + g(n - 1, x)g\left(n - 2, \frac{1}{x}\right)
= g(n - 1, x)\left[g(n - 2, x) + g\left(n - 2, \frac{1}{x}\right)\right]. \tag{14} \]

The initial conditions \( f_1 = f_2 = \{1\} \) specify those on the generating function

\[ g(1, x) = g(2, x) = 1x^1 = x. \tag{15} \]

A nice closed-form expression can be found for \( g(n, x) \):

**Theorem 3.**

\[ g(n, x) = \frac{(1 + x^2)^{F_{n-1}}}{x^{F_{n-2}}}. \tag{16} \]

**Proof.** Simple substitution confirms that this function satisfies the initial conditions (15). We will now show that Eq. (16) is a solution of the recurrence equation (14)

\[
g(n, x) = g(n - 2, x) + g\left(n - 2, \frac{1}{x}\right). \tag{17} \]

The left hand side is simplified as

\[
g(n, x) = \frac{(1 + x^2)^{F_{n-1}}}{x^{F_{n-2}}} \cdot \frac{x^{F_{n-1} - 2}}{x^{F_{n-2}}} = \frac{(1 + x^2)^{F_{n} - F_{n-1}}}{x^{F_{n} - F_{n-1}}}. \tag{18} \]
To evaluate the right hand side of Eq. (17) we first calculate

\[
g(n, \frac{1}{x}) = \left(1 + \frac{1}{x^2}\right)^{F_n - 1} = \left(\frac{1 + x^2}{x^2 - F_n}\right)^{F_n - 1} = \frac{(1 + x^2)^{F_n - 1}}{x^{F_n}}. \tag{19}
\]

Then

\[
g(n - 2, x) + \left(\frac{1}{x}\right)^{1 - 2} = \frac{(1 + x^2)^{F_{n-2} - 1}}{x^{F_{n-2}}} + \frac{(1 + x^2)^{F_{n-2} - 1}}{x^{F_{n-2}}}
= \frac{x^2(1 + x^2)^{F_{n-2} - 1} + (1 + x^2)^{F_{n-2} - 1}}{x^{F_{n-2}}}
= \frac{(1 + x^2)^{F_{n-2}}}{x^{F_{n-2}}}. \tag{20}
\]

The proof is completed by utilizing the identity \( F_n = F_{n-1} + F_{n-2} \).

**Corollary 4.** The generating function \( g(n, x) \) satisfies the functional equation

\[
g(n, x) = x^2 g\left(n, \frac{1}{x}\right). \tag{21}
\]

We can use the generating function formalism to confirm the earlier result on the cardinality of \( f_n \). The size of \( f_n \) is simply the total number of the various integers it contains, and this number is simply the sum of coefficients of the generating function

\[
|f_n| = g(n, 1) = 2^{F_n - 1}. \tag{22}
\]

The generating function \( g(n, x) \) can be written as the binomial expansion

\[
g(n, x) = \sum_{i=0}^{F_n - 1} \binom{F_n - 1}{i} x^{2 - F_n + 2i}. \tag{23}
\]

This provides an explicit description of the set \( f_n \): the elements are \( 2 - F_n + 2i \) with frequency \( \binom{F_n - 1}{i} \), \( i \in [0, F_n - 1] \). Note that this readily provides another proof of Theorem 1.
Few elementary statistical properties of \( f_n \) are supplied by

**Theorem 5.** The average \( \mu \) and variance \( \nu \) of the elements in \( f_n \) are given by

\[
\mu = 1, \\
\nu = F_n - 1.
\]

**Proof.** The average and variance can be computed directly from the generating function \( g(n, x) \) (Wilf [12]). In particular

\[
\mu = \frac{g'(n, x)}{g(n, x)} \bigg|_{x=1},
\]

\[
\nu = \left\{ \frac{d}{dx} \log g(n, x) + \frac{d^2}{dx^2} \log g(n, x) \right\} \bigg|_{x=1}.
\]

Differentiation of the generating function w.r.t. \( x \) yields

\[
g'(n, x) = x^{n-1} - F_n (x^2 + 1)^{n-2} ((x^2 - 1)F_n + 2)
\]

and therefore \( g'(n, x) \big|_{x=1} = 2F_n^{-1} \). Since \( g(n, 1) = 2F_n^{-1} \), \( \mu = 1 \). The logarithmic derivatives are computed as

\[
\frac{d}{dx} \log g(n, x) = \frac{(x^2 - 1)F_n + 2}{x^3 + x},
\]

\[
\frac{d}{dx} \log g(n, x) \bigg|_{x=1} = 1,
\]

\[
\frac{d^2}{dx^2} \log g(n, x) = \frac{(1 + 4x^2 - x^4)F_n - 6x^2 - 2}{(x^3 + x)^2},
\]

\[
\frac{d^2}{dx^2} \log g(n, x) \bigg|_{x=1} = F_n - 2,
\]

and therefore \( \nu = F_n - 1 \).

Finally, we consider the geometric mean of \( f_n \) (more precisely, that of the absolute values of its elements)
where \(|f_n| = 2^{F_n-1}\) is the number of elements in \(f_n\). Table 1 shows the values of the geometric mean for various values of \(n\) and their logarithms.

<table>
<thead>
<tr>
<th>(n)</th>
<th>(G(f_n))</th>
<th>(\ln(G(f_n)))</th>
</tr>
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<tr>
<td>31</td>
<td>615.06</td>
<td>6.42</td>
</tr>
<tr>
<td>34</td>
<td>1265.59</td>
<td>7.14</td>
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<td>37</td>
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</tr>
<tr>
<td>46</td>
<td>22697.3</td>
<td>10.03</td>
</tr>
</tbody>
</table>

Based on Table 1, the growth of \(\ln G(f_n)\) is linear with a slope of approximately 0.24, and therefore the geometric mean of \(f_n\) scales with an exponent of \(\kappa \approx \exp(0.24) \approx 1.271\), i.e.,

\[
\kappa = \lim_{n \to \infty} \frac{1}{n} G(f_n) \approx 1.271. \tag{32}
\]

In other words, the (random) sequence whose \(n\)th element is randomly picked from \(f_n\) exhibits exponential growth characterized by \(\kappa\). Our approximation of \(\kappa\) is very close to that of the square root of the golden ratio \(\phi = \frac{1 + \sqrt{5}}{2}, \sqrt{\phi} \approx 1.272\), the error is just 0.05%. Indeed,

**Theorem 6.**

\[
\kappa = \lim_{n \to \infty} \frac{1}{n} G(f_n) = \sqrt{\phi}. \tag{33}
\]

**Proof.** Introducing \(\alpha = \frac{F_n - 1}{2}\) and \(j = i - \alpha\), the generating function (23) can also be written as
The exponents of the generating functions are the elements of \( f_n \), and the coefficients correspond to their frequencies. Thus the geometric mean of \( f_n \) is computed as

\[
G(f_n) = \prod_{j=-a}^{a} \left( 2j + 1 \right)^{2a} \left( a + j \right)^{4a} / a
\]

Since \( \sum_{j=-a}^{a} \frac{1}{4a} \left( a + j \right) = 1 \), the summand in (35) describes a weighted binomial distribution. In the \( a \to \infty \) limit the binomial distribution can be well approximated by the continuous normal distribution (Weisstein [11])

\[
\frac{1}{4a} \left( a + j \right) \sim \frac{1}{\sqrt{\pi}a} \exp \left( - \frac{j^2}{a} \right).
\]

The sum is approximated as

\[
\sum_{j=-a}^{a} \frac{1}{4a} \left( a + j \right) \log |2j + 1| \sim \sum_{j=-a}^{a} \frac{1}{4a} \left( a + j \right) \log |2j|
\]

\[
= \log 2 + \sum_{j=-a}^{a} \frac{1}{4a} \left( a + j \right) \log |j| \sim \log 2 + \frac{1}{\sqrt{\pi}a} \int_{-a}^{a} \exp \left( - \frac{j^2}{a} \right) \log |j| \, dj
\]

\[
\sim \log 2 + \frac{1}{\sqrt{\pi}a} \int_{-\infty}^{\infty} \exp \left( - \frac{j^2}{a} \right) \log |j| \, dj = \log 2 + \frac{2}{\sqrt{\pi}a} \int_{0}^{\infty} \exp \left( - \frac{j^2}{a} \right) \log j \, dj.
\]

Since

\[
\int_{0}^{\infty} \exp \left( - \frac{j^2}{a} \right) \log j \, dj = \frac{\sqrt{\alpha \pi}}{4} (\log \alpha - 2 \log 2 - \gamma),
\]
we obtain

\[ \log G(f_n) = \sum_{j=-a}^{a} \frac{1}{4^a} \left( \frac{2a}{a + j} \right) \log|2j + 1| \sim \frac{\log a - \gamma}{2}. \]  

(39)

Here \( \gamma \approx 0.5772 \) is the Euler-Mascheroni constant. Finally,

\[ \frac{1}{n} \log G(f_n) \sim \frac{\log a}{2n} \sim \frac{\log \left( \frac{F_n - 1}{2} \right)}{2n} \sim \frac{\log \phi}{2} = \log \sqrt{\phi} \]  

(40)

and thus

\[ \lim_{n \to \infty} \frac{1}{n} G(f_n) = e^{\log \sqrt{\phi}} = \sqrt{\phi} = \kappa. \]  

(41)

4. Conclusions

Motivated by studies on the random Fibonacci sequence we defined and investigated a recurrence whose elements are multisets. A closed-form generating function is derived to explicitly characterize the elements of these multisets and their exact averages and variances. The geometric mean of the multisets is proved to grow exponentially, scaling with the square root of the golden ratio.

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References


