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Interior Parameters, Exterior Parameters, and a Cayley-Like Transform

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Introduction

ARGUABLY, the most fundamental representation for describing the relative orientation of a rigid body is the orientation matrix, also called the direction cosine matrix. Few believe, however, that this matrix provides the most convenient parameterization of orientation. For that, many instead turn to one of a collection of three-parameter sets, even though it is known that every three-parameter orientation set will encounter some kind of singularity, a fact shown by Stuelpnagel [1]. Others turn to the Euler parameters, which are a nonsingular four-parameter set.

The direct relationship between each of the various parameter sets and the orientation matrix is considered to be of basic importance. So, too, is the time derivative of such relationships, although the angular velocity vector or matrix is usually substituted for the time derivative of the orientation matrix. The purpose of this note is to investigate the direct relationship between a particular three-parameter orientation set, viz., the modified Rodrigues parameters (MRP), and the orientation matrix. The modified Rodrigues parameters were originally developed by Wiener [2], rediscovered by Marandi and Modi [3], and investigated quite fully by Tsiotras and Longuski [4] and Schaub and Junkins [5]. Nevertheless, there is more to say.

One Way to Describe Orientation

The relative orientation of one body to another in three-dimensional space can be measured with an orthogonal matrix. This matrix, C, can be parameterized in a number of ways, and one popular choice during the last decade has been the modified Rodrigues parameters:

\[
C = I + \frac{8S^2 - 4(1 - s^2)S}{(1 + s^2)^2} \tag{1}
\]

Here, S is a skew-symmetric matrix arrangement of the elements of the modified Rodrigues parameters (\(s_1 = S_{32}, s_2 = S_{13}, s_3 = S_{21}\)) and \(s^2 = S_{11} + S_{22} + S_{33}\). The modified Rodrigues parameters provide a valid description of orientation from 0 deg to something just shy of 360 deg (the angular measures discussed in this note are taken to mean Euler’s principal angle). At precisely 360 deg, the modified Rodrigues parameters become infinite, and so we say that they exhibit an orientation singularity at this configuration. This singularity condition is exposed if one uses Euler’s principal line and angle to introduce the modified Rodrigues parameters in the standard way: \(s_i = \tan(\phi/4)\epsilon_i\), for \(i = 1, 2, 3\), where \(\epsilon_i\) are the elements of Euler’s principal line and \(\phi\) is Euler’s principal angle.

It is often stated that the best way to determine the modified Rodrigues parameters when given an orientation matrix C is to take an indirect path through the Euler parameters. Actually, there is a more straightforward way:

\[
S = \frac{1}{(\sqrt{\xi + 1} + 2)(\sqrt{\xi + 1})}(C^T - C) \tag{2}
\]

Here, \(\xi\) is the trace of C. This equation appears to be a new result and can be verified by the substitutionary use of Eq. (1). To clarify, using Eq. (1) in Eq. (2) relates S to a fraction of itself:

\[
S = \frac{8(1 - s^2)}{(1 + s^2)^2}((\sqrt{\xi + 1} + 2)(\sqrt{\xi + 1})S \tag{3}
\]

But the direct calculation of the trace of C from Eq. (1) shows \(\xi = (3s^2 - 1)(s^2 - 3)/(1 + s^2)^2\), which allows \(\sqrt{\xi + 1} = 2(1 - s^2)/(1 + s^2)\). This result reduces the coefficient on the right side to one, and so Eqs. (1) and (2) are compatible.

Note that Eq. (2) only encounters problems at an orientation equal to 180 deg. At this configuration, \(\xi = -1\) whereas \(C = C^T\). Consequently, the right side of Eq. (2) gives a 0/0 condition, in which the numerator is actually a zero matrix. This is related to the fact that C as a function of the modified Rodrigues parameters is not injective at 180 deg if the domain of the modified Rodrigues parameters is taken to include positive and negative values. For configurations of 180 deg, though, the modified Rodrigues parameters become the elements of Euler’s principal line; thus, the modified Rodrigues parameters can be reclaimed from Eq. (27) in Sec. 2.1 of Hughes [6].

A new algorithm for finding the modified Rodrigues parameters from the orientation matrix C is as follows:

1) Compute \(\xi = \text{trace of } C\).
2) If \(\xi = -1\), compute the modified Rodrigues parameters from Eq. (27) in Sec. 2.1 of Hughes (see the Appendix at the end of this note.)
3) Else, compute the modified Rodrigues parameters from Eq. (2).

Interior and Exterior Parameters

Remarkably, Eq. (2) always produces modified Rodrigues parameters whose magnitude is bounded above by one. This truth follows from relating Eq. (2) to the expression that relates Euler’s principal line to the elements of C. To clarify, one can rewrite Eq. (2) using Eq. (24) in Sec. 2.1 of Hughes [6], which is given here as:

\[
E = \frac{2\sin \phi}{(\sqrt{\tau + 1} + 2)(\sqrt{\tau + 1})}E \tag{4}
\]

\[
\sin \phi = \frac{\sqrt{\cos \phi + 1} + \sqrt{2}(\sqrt{\cos \phi + 1})}{E}
\]

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Excluding configurations of 180 deg, this expression illustrates that the magnitude of the modified Rodrigues parameters is bounded above by one because Euler’s principal line has unit length and the factor multiplying $E$ is bounded between $\pm 1$. The 180 deg configuration can be analyzed by using some half-angle identities (viz., $\cos \phi = 2\cos^2(\phi/2) - 1$ and $\sin \phi = 2\sin(\phi/2)\cos(\phi/2)$) to rewrite the aforementioned factor as $\tan(\phi/4)$, which equals one at 180 deg. Consequently, Eq. (2) always produces modified Rodrigues parameters whose magnitude is bounded above by one.

To appreciate the significance of this result, note that there is a “shadow” set of modified Rodrigues parameters, distinct from $S$, that represents the same relative orientation of one body to another. This shadow set, $S_s$, is like a nonidentical twin to $S$, and either can rightfully claim to be modified Rodrigues parameters representing orientation: both satisfy Eq. (1). The character of $S$ and $S_s$ is such that, when one set is within a unit sphere, the other set is outside. At 180 deg, however, both sets are on the unit sphere.

Because Eq. (2) always produces modified Rodrigues parameters within the closed unit sphere, the corresponding shadow set will always be on the closed exterior of the unit sphere. These shadow parameters (SP) are related to the orientation matrix $C$ by an expression that is strikingly similar to Eq. (2):

$$S_s = \frac{1}{(\sqrt{\zeta} + 1 - 2/(\sqrt{\zeta} + 1))}(C^T - C)$$

This equation can be verified by substitutionary use of Eq. (1). To clarify, using Eq. (1) in Eq. (3) relates $S_s$ to a fraction of itself:

$$S_s = \frac{8(1 - s^2)}{(1 + s^2)^2(\sqrt{\zeta} + 1 - 2/(\sqrt{\zeta} + 1))} S_s$$

But the direct calculation of the trace of $C$ from Eq. (1) allows $\sqrt{\zeta} + 1 = -2(1 - s^2)/(1 + s^2)$. This reduces the coefficient on the right side to one; therefore, Eqs. (1) and (2) are compatible.

It is not surprising that Eq. (3) has the same problem as Eq. (2) at 180 deg configurations. But Eq. (3) has an additional issue, which occurs at orientations equal to 0 or 360 deg where $\zeta = 3$. Truthfully, this new issue is not surprising either, as it simply reflects the fact that, when one set of modified Rodrigues parameters equal zero, the other set of modified Rodrigues parameters are infinite [2].

The features surrounding Eqs. (2) and (3) encourage one to repartition the collection of modified Rodrigues parameters from the traditional modified Rodrigues parameters and their shadow set to interior and exterior modified Rodrigues parameters. Or perhaps even more simply stated, the interior parameters (IP) are $S$ and the exterior parameters (EP) are $S_s$. This is graphically shown in Fig. 1 using the concept of stereographic projection.

**Cayley-Like Transform**

Besides Eqs. (1–3), there are other relationships between orientation matrices and skew-symmetric matrices. The Cayley transform in three-dimensional space relates an orthogonal orientation matrix $C$ to a skew-symmetric matrix $Q$. The three elements of $Q$ are the classic Rodrigues parameters. The transforms, from $C$ to $Q$ and back again, are at once elegant and unforgettable:

$$C = (I + Q)^{-1}(I - Q) = (I - Q)(I + Q)^{-1}$$

$$Q = (I + C)^{-1}(I - C) = (I - C)(I + C)^{-1}$$

A Cayley-like transform exists for the interior and exterior modified Rodrigues parameters. *Interior parameters:*

$$C = (kI + S_s)^{-1}(kI - S_s) = (kI - S_s)(kI + S_s)^{-1}$$

$$S = k(I + C)^{-1}(I - C) = k(I - C)(I + C)^{-1}$$

**Fig. 1** A common graphical tool for presenting the modified Rodrigues parameters is stereographic projection, wherein a point on the unit sphere is mapped to the projection plane by connecting the aforementioned point to a projection point. Using this device, the modified Rodrigues parameters are partitioned into the following: a) traditional modified Rodrigues parameters and their shadow set, and b) new partitioning comprises interior and exterior parameters.

This set of transformations appears to be a new result and can be verified by substitutionary use of Eq. (1). (This is demonstrated in the Appendix.) The parameter $k$ in these equations is a scalar. In Eq. (6), $k$ depends on the elements of the interior modified Rodrigues parameters, $k(s) = (1 - s^2)/2$, whereas in Eq. (7) it depends on the elements of the orientation matrix, $k(C) = (\sqrt{\zeta} + 1)/(\sqrt{\zeta} + 1)$. We have the same issues, for the same reasons, at orientations corresponding to 180 deg. At 180 deg, $s = 1$, and so Eq. (6) is replaced with $C = 1 + 2S$ [see Eq. (1)]. Likewise, $C$ has an eigenvalue at $-1$, which means det$(1 + C) = 0$, and so Eq. (7) is replaced with $C = (I + C)^{-1}$ in Sec. 2.1 of Hughes.

**Exterior parameters:**

$$C = (kI + S_s)^{-1}(kI - S_s) = (kI - S_s)(kI + S_s)^{-1}$$

$$S_s = k(I + C)^{-1}(I - C) = k(I - C)(I + C)^{-1}$$

Similar to before, $k$ in Eq. (8) depends on the elements of the exterior modified Rodrigues parameters, $k(S_s) = (1 - S_s^2)/2$; in Eq. (9) it depends on the elements of the orientation matrix, $k(C) = (\sqrt{\zeta} + 1)/(\sqrt{\zeta} + 1)$. Note that $S_s$ is taken to mean the magnitude of the exterior modified Rodrigues parameters.

**Discussion**

There is no orientation that the interior parameters cannot describe. They are a global (i.e., defined everywhere), three-parameter set that is two valued at orientations corresponding to 180 deg. Moreover, because their magnitude is bounded by one, the interior parameters naturally represent the shortest distance back to the origin, which is taken to mean the minimum angular distance and is known to be Euler’s principal angle (see, for example, problem 2.20 in [6] or [8]). The interior parameters are thus linked to the best or optimal solution to a decision problem (i.e., “What is the minimum angular distance between two elements of SO(3)?”) and therefore reflect all the associated virtues of such solutions. The traditional modified Rodrigues parameters and their shadow set can be patched together to represent the same thing.

Every continuous three-parameter orientation set will suffer from one of two types of singularity [1]. Either the kinematic differential
equations that govern their evolution will be singular, which means there is a point \((t_i, x_i)\) for the equation \(\dot{x} = f(t, x)\) where \(f\) and \(\frac{\partial f}{\partial x}\) are not finite and continuous functions at every point \((t, x)\) inside an open domain \(D\) containing \((t_i, x_i)\) (see pp. 11–13 in [2]), or there will be particular configurations for which the parameters are undefined. This is not distinction without difference. The first case describes singularities of the differential equations whereas the second case describes singularities of their solutions. The first case is commonly called a kinematic singularity, and this happens when an orientation generates nonunique parameter values for a continuous three-parameter set; the second case is commonly called an orientation singularity, and (as already stated) this happens when the parameters of a continuous set become undefined. This is not distinction without difference. The Euler angles are a common example of a three-parameter set that exhibits a kinematic singularity, whereas the modified Rodrigues parameters are a common example of a parameter set that exhibits an orientation singularity. Critically, parameters that exhibit orientation singularities can display difficulties even in the case of static displacement, whereas parameters that exhibit kinematic singularities have no such problems.

Based on the preceding paragraph, the nonuniqueness of the interior parameters at 180 deg suggests that they will exhibit a kinematic singularity at this configuration. The normal rules do not apply, however, because the interior parameters are not a continuous three-parameter orientation set. A discontinuity occurs at 180 deg, and this can be seen in the relationship between the interior parameters and Euler’s principal line and angle. The relationship was listed earlier in a skew-symmetric matrix form, but it is listed here in component form: \(x_i = e_i \sin \phi / (\sqrt{\cos \phi + 1} + \sqrt{2}) / (\sqrt{\cos \phi + 1})\). For the illustrative case of an eigenaxis rotation, wherein Euler’s principal line remains fixed while the principal angle varies, it is evident that, as the function multiplying the \(e_i\) components goes, so go the interior parameters \(x_i\). Analysis shows that this function is discontinuous at 180 deg while having a finite derivative that equals one-half.

Interestingly, the interior parameters seem fundamentally different than the standard modified Rodrigues parameters. The modified Rodrigues parameters exhibit an orientation singularity, whereas the interior parameters do not; and the modified Rodrigues parameters are a continuous orientation set, whereas the interior parameters are not. A graphical comparison of the troubles encountered by the various parameters is presented in Fig. 2. The exterior parameters fair worst of all because they exhibit an orientation singularity at configurations corresponding to 0 and 360 deg and a jump discontinuity at 180 deg. Figure 3 shows the parameters as they correspond to a single-axis rotation. The orientation singularities and jump discontinuities are evident:

\[
\text{IP} = \frac{\sin \phi}{\sqrt{2(\cos \phi + 1) + \cos \phi + 1}}
\]

\[
\text{EP} = \frac{\sin \phi}{-\sqrt{2(\cos \phi + 1) + \cos \phi + 1}}
\]

\[
\text{MRP} = \tan(\phi/4)
\]

\[
\text{SP} = \tan(\phi - 2\pi)/4
\]

The standard modified Rodrigues parameters and their shadow set and the interior and exterior parameters all satisfy Eq. (1). Consequently, we expect the same kinematic equations to rule them all. The kinematic differential equations for any parameter set can be directly constructed from \(C\) using Poisson’s kinematical equation (see p. 53 in [10]), which relates the time derivative of \(C\) to the skew-symmetric angular velocity matrix \(\Omega\):

\[
\dot{C} = -\Omega C \Rightarrow \Omega = -\dot{C}C^T \tag{10}
\]

Equation (1) essentially relates \(C\) to the elements \(s_j\); therefore, Eq. (10) can be rewritten using the chain rule of calculus:

\[
\Omega_{ij} = -[(\partial C_{ia}/\partial s_j)C_{ja}, (\partial C_{ia}/\partial s_j)C_{ja}, (\partial C_{ia}/\partial s_j)C_{ja}][\dot{s}_1, \dot{s}_2, \dot{s}_3]^T \tag{11}
\]

An index summation convention is implied here, which means that the repeated index \(n\) is summed over its range from 1 to 3. The components of the angular velocity matrix depend on the components of the angular velocity vector: \(\Omega_{12} = -\omega_1\), \(\Omega_{13} = -\omega_2\), and \(\Omega_{23} = -\omega_3\). This allows Eq. (11) to be written in terms of the angular velocity vector components, \(\omega_i = B_i\hat{s}_i\), where summation on \(j\) is implied:

\[
B = \begin{bmatrix}
(\partial C_{2a_i}/\partial s_1)C_{ja}, & (\partial C_{2a_i}/\partial s_2)C_{ja}, & (\partial C_{2a_i}/\partial s_3)C_{ja}
(\partial C_{3a_i}/\partial s_1)C_{ja}, & (\partial C_{3a_i}/\partial s_2)C_{ja}, & (\partial C_{3a_i}/\partial s_3)C_{ja}
(\partial C_{\text{int}}/\partial s_1)C_{ja}, & (\partial C_{\text{int}}/\partial s_2)C_{ja}, & (\partial C_{\text{int}}/\partial s_3)C_{ja}
\end{bmatrix} \tag{12}
\]

The explicit partial derivatives in Eq. (12) can be efficiently computed and simplified using symbolic mathematical software. The final result is the familiar matrix expression that relates the time derivative of the modified Rodrigues parameters to the angular velocity components:

\[
B = (4/(1 + s^2))(1 + s^2)I + 2SS - 2S \tag{13}
\]

An uncommon form can be obtained by using some matrix algebra manipulations:
Some Relations to Previous Work

Although Eq. (2) is the first natural definition of the interior parameters, many authors have used them extensively in their attitude and control work. Indeed, on p. 25 of Schaub’s dissertation [11], he states, “Referring to the MRPs from here on, it will be understood that the combined set of original and shadow MRPs is meant.” In the sentences before this statement, he suggests an artificial switching between the modified Rodrigues parameters and their shadow set at the unit sphere surface. The successful use of the interior parameters is discussed in Refs. [21–26] in [11] (e.g., see the work of Schaub et al. [12]).

Schaub et al. [13] and Tsiotras et al. [14] have introduced a Cayley transform between an orientation matrix and the modified Rodrigues parameters. Unlike ours, however, their direct transform is actually one way, and it is only two way through an indirect path. That is, they show that the orientation matrix is quadratically related to the modified Rodrigues parameters, but they are unable to give a direct transformation from the orientation matrix back to the modified Rodrigues parameters. Thus, to go from the orientation matrix back to the modified Rodrigues parameters, they write $C$ as the square of a second orthogonal matrix, $W$, and it is through $W$ that the modified Rodrigues parameters are related to $C$. Their approach takes advantage of the fact that the modified Rodrigues parameters of $C$ are exactly the classic Rodrigues parameters of $W$. Remarkably, their mapping holds in higher dimensions, whereas the Cayley-like transforms mentioned in this article do not. Our Cayley-like work is their Cayley transform only in three dimensions, where $S^3 = -s^2S$. This can be demonstrated by equating our Cayley expression for $C$ to their Cayley expression for $C$:

\[
(kI - S)(kI + S)^{-1} = C = (I + S)^{-2}(I - S)^2
\]

Expanding both sides while using $k = (1 - s^2)/2$ confirms the equality.

The orthogonal matrix $W$ appearing in Eq. (15) is the same $W$ that appears in the Cayley transform work of Schaub et al. [13] and Tsiotras et al. [14], and Eq. (14) appears on p. 486 of Shuster’s work [7] although no proof is given. Our Eq. (15) ties Shuster to Schaub et al. [13] for the first time.

Similar to the interior parameters, the Argyris parameters, discussed by Hassemplug [15], are another globally defined, three-parameter set of orientation parameters. Hughes happens to call these the “axis/angle variables,” and they are also known as elements of the “Euler vector.” These parameters are two valued at an orientation corresponding to 0 deg; consequently, their kinematic behavior is singular at this orientation. One advantage of the interior parameters over the Argyris parameters is that the interior parameters automatically represent the shortest distance back to the origin.

Conclusions

Extracting the modified Rodrigues parameters directly from the orientation matrix leads to a new categorization that is different from the usual modified Rodrigues parameters and their shadow set. The new categorization is composed of a useful set of parameters, the interior parameters, which possess a jump discontinuity at 180 deg, and a seemingly useless set of parameters, the exterior parameters, which possess multiple orientation singularities in addition to a jump discontinuity. This approach leads to a new algorithm for calculating the interior parameters from the orientation matrix without using the Euler parameters, which are typically used. Moreover, the kinematic differential equations can be constructed directly from the orientation matrix and its time derivative without relying on the Euler parameters. The approach is general and can be applied to any orientation parameter set. Furthermore, the new relationship between the interior parameters and the orientation matrix can be concisely written as a Cayley-like transform.

The interior parameters are able to conveniently describe all orientations. Their magnitude stays within the closed unit sphere; conversely, the magnitude of the exterior parameters remains outside a unit sphere. Also note that the interior parameters are associated with the interior angular measure from the origin, whereas the exterior parameters are associated with the exterior angular measure. Therefore, the interior and exterior parameters could be suitably called the acute and obtuse parameters.

Appendix: Equations in Hughes and Cayley-Like Transformations

Equation (27) in Sec. 2.1 of Hughes is as follows:

\[
s_i = \pm \sqrt{(1 + C_{ii})/2} \quad \text{for } i = 1, 2, 3
\]

\[
s_{ij} = C_{ij}/2 \quad \text{for the sets } (i, j) = (1, 2), (2, 3), (3, 1)
\]

According to Hughes, the last three equations resolve the sign ambiguities in the first three equations.

As for the Cayley-like transformations, Eq. (7) can be verified by the substitutionary use of Eq. (1). To clarify, using Eq. (1) to replace $I + C$ and $(I - C)$ in a rearranged form of the right-most expression in Eq. (7) produces linear and quadratic terms in $S$:

\[
2 \left( \frac{8s^2}{k(1 + s^2)^2} \right) S - \frac{4(1 - s^2)}{k(1 + s^2)^2} S^2 = \frac{4(1 - s^2)}{k(1 + s^2)^2} S - \frac{8}{(1 + s^2)^2} S^2
\]

Here, $S^3 = -s^2S$ was used on the left side. The subsequent use of $k = (1 - s^2)/2$ confirms the equality of this expression, which indicates the compatibility of Eqs. (1) and the right-most part of (7).

Having validated the right-most part of Eq. (7), the other expressions in Eqs. (6) and (7) can be established using straightforward matrix algebra manipulations (see, for example, pp. 53–56 of Junkins and Kim [16]). Typically, these matrix algebra manipulations involve nothing more than the properties of the matrix transpose of a sum and product, together with the skew-symmetry of $S$ and the orthogonality of $C$.

1) From $S = k(I - C)(I + C)^{-1}$ to $S = k(I + C)^{-1}(I - C)$:

\[
S = k(I - C)(I + C)^{-1} \Rightarrow S(I + C) = k(I - C)
\]

\[
\Rightarrow -C(S^T) S = k(I - C^T)
\]

\[
\Rightarrow -(C + I) S = k(C - I)
\]

\[
\Rightarrow S = k(I + C)^{-1}(I - C)
\]

2) From $S = k(I + C)^{-1}(I - C)$ to $C = (kI - S)(kI + S)^{-1}$:
\[ S = k(I + C)^{-1}(I - C) \Rightarrow (I + C)S = k(I - C) \]
\[ \Rightarrow CS + kC = kI - S \]
\[ \Rightarrow C(S + kI) = (kI - S) \]
\[ \Rightarrow C = (kI - S)(kI + S)^{-1} \]

3) From \( C = (kI - S)(kI + S)^{-1} \) to \( C = (kI + S)^{-1}(kI - S) \):

This demonstration of item 3 is like 1 above, mutatis mutandis.

References


