Elements of Spacecraft Control

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A complete set of lecture notes for a one-semester, graduate-level course in control for spacecraft models. Topics include a review of rigid body mechanics; review of stability concepts and Lyapunov analysis; Lyapunov control methodology; methods for rigid body regulation involving quaternions and other Euler-parameter types of attitude variables; basics of reference motion tracking and reference motion trajectories for rotational motion; reference motion tracking without rate information; adaptive control for rigid body rotational motion; gyrostat models and control for nutation damping & flat-spin recovery; models for single gimbal control moment gyros; velocity- and acceleration-based steering laws; and feedback control singularities and null motion feedback control.
Preface. This book is my collection of topics for a graduate-level course on introductory spacecraft control. My preferences in content and style are reflected in the material and its presentation (I have continued to use my 1-page, 1-topic format), and in the nuances when contrasted with more familiar arrangements.

My focus is on the underlying principles of spacecraft control, therefore much of the tedious algebra and calculus from one equation to the next is left for the reader.

JEH

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For Antonio, Rosario, Carlos, and Guadalupe
References and Supplemental Sources.


Junkins, J.L. & Kim, Y. 1993 *Introduction to Dynamics and Control of Flexible Structures*. Washington, DC: AIAA.


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0 Notation
A Strict View. One commonly encounters three entities when discussing the principles of mechanics: vectors and tensors; column and matrix arrangements of vectors and tensors; and groupings of equations. When dealing with these objects, a precise notation can help clarify concepts and developments. One set of useful notations is given below.

1. Scalars are written in math italics type.

   scalar: \( A \) or \( a \)

2. Physical vectors and tensors are written in boldface type.

   vector: \( \mathbf{v} \)
   tensor: \( \mathbf{T} \)

3. A specific arrangement of the components of a physical vector or tensor in a particular reference frame are written using brackets around the boldface symbol with an outside subscript that indicates the reference frame. These are sometimes called list vectors or component matrices.

   column arrangement of the components of a vector: \([\mathbf{v}]_b\)
   matrix arrangement of the components of a tensor: \([\mathbf{T}]_b\)

4. A set of equations can sometimes be conveniently written in a collected form using rectangular matrices. Also, rectangular matrices can be used to write a collection of variables or parameters. These matrices can have nothing to do with the components of a vector or tensor in a particular reference frame. Consequently, boldface type is not needed but brackets are used to indicate the matrix.

   rectangular arrangement: \([A]\) or \([x]\)

These notations were stringently followed in *Kinematic and Kinetic Principles*, but we’ll take a more relaxed position in these notes.
A Relaxed View. A precise notation is most needed in dynamics when dealing with rotational motions and multiple reference frames. When there is little chance of confusion (for example, when only one reference frame is involved), some relaxed versions of the previously mentioned strict notations can be used.

1. Scalars are still written in math italics type.

   scalar: \( A \) or \( a \)

2. Physical vectors, tensors, and arrangements of their components in a particular reference frame (i.e., their associated list vectors or component matrices) are written using only boldface type. The change here is that the surrounding brackets and outside subscripts are omitted when writing a component matrix.

   physical vector or list vector or component column: \( \mathbf{v} \)

   physical tensor or list tensor or component matrix: \( \mathbf{T} \)

3. Rectangular matrices that represent a collection of variables, parameters, or constants are written using boldface type even though these matrices can have nothing to do with the components of a vector or tensor in a particular reference frame. The change here is that the surrounding brackets are omitted and the font is changed to boldface.

   rectangular arrangement: \( \mathbf{A} \) or \( \mathbf{x} \)
1 Point Mass & Rigid Body Equations
An Introduction. The purpose of this first chapter is to refresh our understandings of dynamic models and forms of the governing equations. Notationally, the relaxed view is used. That is, physical vectors, their component matrix representations, and ordinary rectangular matrices that represent a collection of variables, parameters, or constants are written using boldface type with no surrounding brackets. Nearly everything becomes, for example, $A$ or $x$. 
Common Modeling. A physical system is commonly modeled as a point mass, or a rigid body, or a flexible body, or some combination thereof. Although these models are truly idealizations and each has its limitations, they nevertheless provide a fair representation of observed behavior.

A point mass model can undergo translational motion and therefore this representation can have up to three degrees of freedom. The fundamental law that dictates the motion is Newton’s second law of motion, \( f = ma \).

A rigid body model can undergo rotational motion in addition to translational motion. Consequently, this representation has up to six degrees of freedom. The fundamental laws that dictate the motion are Newton’s and Euler’s laws of motion, \( f = ma_c \) and \( \ell_c = \dot{\ell}_c \). A dot over a vector means the inertial derivative, i.e., the time derivative as seen by an inertial observer.

A flexible body model allows small deformations beyond large translational and rotational rigid body motions. The small deformations amount to an infinite number of degrees of freedom. Strictly speaking, Newton’s and Euler’s laws of motion still rule, but the governing equations of motion are commonly derived using Hamilton’s Integral Principle (also known as the Generalized Hamilton’s Principle), \( \int_{t_1}^{t_2} (\delta L + \delta W) \, dt = 0 \).

Each of the three modeling techniques can be taken a step further to result in a linear system model. The linear system model is usually written in mechanical second-order form.

\[
M \ddot{x} + C \dot{x} + K x = 0 \quad (6.1)
\]

A state space form of a linear system model is sometimes preferable, wherein the position and velocity coordinates are stacked.

\[
\dot{\sigma} = A \sigma \quad \text{where} \quad \sigma = [x, \dot{x}]^T \quad (6.2)
\]
Point Mass Basics. A point mass model can undergo translational motion, and the fundamental law that governs the motion is \( N2L, f = ma \). This expression generates second-order, ordinary differential equations. A procedure for generating the specific equations can be established: identify the system; draw a free body diagram (FBD); establish relevant reference frames; write a vector representation of the forces; perform the vector kinematics up to the acceleration level; and use \( N2L \) to determine the governing equations of motion.

A classic example is a point mass in a rotating tube. One end of the tube is fixed, and the tube rotates in a horizontal plane at a known constant rate \( \dot{\theta} \). The mass is free to move within the rotating tube and its location is given by the coordinate \( r \). The system is the point mass. A proper FBD will include the force due to the gravity field, which acts along the \( -\hat{b}_3 \) direction, the contact force of the side wall touching the point mass, which acts along the \( \hat{b}_2 \) direction, and the contact force from the bottom of the tube, which acts along the \( \hat{b}_3 \) direction. The \( b^+ \) frame rotates with the tube and \( n^+ \) is taken to be an inertial frame. The \( \hat{n}_3 \) and \( \hat{b}_3 \) axes are collinear. The inertial acceleration vector is \( a = (\ddot{r} - r\dot{\theta}^2)\hat{b}_1 + 2\dot{r}\dot{\theta}\hat{b}_2 \). Newton’s equation of motion for a point mass gives three scalar equations involving the time evolution of the radial position of the point mass and expressions for the side wall and bottom contact forces.

\[
m(\ddot{r} - \dot{r}\dot{\theta}^2) = 0 \quad ; \quad 2m\dot{r}\dot{\theta} = f_w \quad ; \quad 0 = f_n - mg \quad (7.1)
\]
Rigid Body Dynamics. Rigid body motion has six degrees of freedom and so a minimum of twelve first-order, ordinary differential equations are necessary to completely describe the motion.

Six equations are used to describe the translational motion of the mass center, which acts like a “super particle”.

\[
ma_c = m\dot{v}_c = \mathbf{f} \quad ; \quad \dot{r}_c = \mathbf{v}_c \quad (8.1)
\]

The overdot on the vectors means the inertial time deriviative.

Six different equations are used to describe the rotational motion about the mass center.

\[
\dot{\mathbf{h}}_c = \ell_c \quad ; \quad \dot{\mathbf{\mu}} = A\mathbf{\omega} \quad (8.2)
\]

The vector \( \mathbf{\omega} \) is the angular velocity in a body-fixed frame. The dynamics portion, the vector equation on the left, is Euler’s law for rotational motion, \( \text{E2L}_c \). Euler’s famous rotational equations are created by componentizing this law in a body-fixed, principal axes frame.

\[
\begin{align*}
I_1 \dot{\omega}_1 + (I_3 - I_2)\omega_2 \omega_3 &= \ell_1 \\
I_2 \dot{\omega}_2 + (I_1 - I_3)\omega_1 \omega_3 &= \ell_2 \\
I_3 \dot{\omega}_3 + (I_2 - I_1)\omega_1 \omega_2 &= \ell_3 
\end{align*}
\quad (8.3)
\]

The kinematics portion of rotational motion, i.e., the matrix equation on the right-most part of eq. (8.2), is needed to complete the attitude description. The elements of the column matrix \( \dot{\mathbf{\mu}} \) represents the time derivative of a collection of orientation parameters. In most applications, the orientation parameters are taken to be a set of Euler angles, or the classic Rodrigues parameters, or the modified Rodrigues parameters, or the Euler parameters.

The next few pages will present the classic Rodrigues parameters, the modified Rodrigues parameters, and the Euler parameters.
The Classic Rodrigues Parameters. The classical Rodrigues vector and its elements, the classical Rodrigues parameters (CRPs), are one choice for parameterizing relative orientation. Kinematically speaking, the important connections include the relationships between the CRPs and the orientation matrix, and the time derivative of these relationships, although the angular velocity vector or matrix is usually substituted for the time derivative of the orientation matrix.

It is implicitly understood that, when relevant, the CRP vector $\mathbf{q}$ and the angular velocity vector $\boldsymbol{\omega}$ are componentized in a body-fixed frame $b^+$ in the following expressions. Also, as a shorthand device we take $\mathbf{q}^2 \equiv q_1^2 + q_2^2 + q_3^2 = ||\mathbf{q}||^2$. In the equations below, $\mathbf{q}^\times$ means a skew-symmetric matrix arrangements of the elements of $\mathbf{q}$.

**Orientation**

$$C = 1 - \frac{2}{1 + q^2} \mathbf{q}^\times (1 - \mathbf{q}^\times) \quad (9.1)$$

$$\mathbf{q}^\times = \frac{1}{\varsigma + 1} \left( C^\top - C \right) \quad (9.2)$$

where, $\varsigma$ is the trace of $C$

**Kinematics**

$$\dot{\mathbf{q}} = A\boldsymbol{\omega} \quad ; \quad A = \frac{1}{2} \left( 1 + \mathbf{q}^\times + \mathbf{q}\mathbf{q}^\top \right) \quad (9.3)$$

$$\boldsymbol{\omega} = B\dot{\mathbf{q}} \quad ; \quad B = \frac{2}{1 + q^2} \left( 1 - \mathbf{q}^\times \right) \quad (9.4)$$

The classical Rodrigues vector $\mathbf{q}$ is undefined when Euler’s principal angle is 180 deg. We say that the classical Rodrigues vector is singular at this orientation, so this parameterization exhibits an orientation singularity.
The Modified Rodrigues Parameters. The *modified Rodrigues vector* and its elements, the modified Rodrigues parameters (MRPs), are one choice for parameterizing relative orientation. Like the CRPs, the important connections include the relationships between the MRPs and the orientation matrix, and the time derivative of these relationships, although the angular velocity vector or matrix is usually substituted for the time derivative of the orientation matrix.

It is implicitly understood that, when relevant, the MRP vector $s$ and the angular velocity vector $\omega$ are componentized in a body-fixed frame $b^+$ in the following expressions. Also, as a shorthand device we take $s^2 \equiv s_1^2 + s_2^2 + s_3^2 = ||s||^2$.

**Orientation**

$$C = 1 + \frac{8s^\times s^\times - 4(1 - s^2)s^\times}{(1 + s^2)^2}$$  \hspace{1cm} (10.1)

$$s^\times = \frac{1}{(\sqrt{\zeta + 1} + 2)(\sqrt{\zeta + 1})} \left( C^T - C \right)$$ \hspace{1cm} (10.2)

where, $\zeta$ is the trace of $C$

**Kinematics**

$$\dot{s} = A\omega \quad ; \quad A = \frac{1}{4} \left( (1 + s^2)I + 2s^\times s^\times + 2s^\times \right)$$  \hspace{1cm} (10.3)

$$\omega = B\dot{s} \quad ; \quad B = \frac{4}{(1 + s^2)^2} \left( (1 + s^2)I + 2s^\times s^\times - 2s^\times \right)$$ \hspace{1cm} (10.4)

The above descriptions of the modified Rodrigues vector $s$ leads to a discontinuity when Euler’s principal angle is precisely 180 deg.\(^1\)

An uncommon but elegant form for $A$ and $B$ can be obtained by using some matrix algebra manipulations. This shows the amazing truth that these matrices are nearly orthogonal.

$$A = \frac{1 + s^2}{4} W^T \quad ; \quad B = \frac{4}{1 + s^2} W$$ \hspace{1cm} (10.5)

where $WW^T = I$ and $WW = C$

---

The Euler Parameters. Unlike the CRPs and MRPs, the Euler parameters (EPs) are a four-parameter set for describing relative orientation. Here, too, the important connections include the relationships between the EPs and the orientation matrix, and the time derivative of these relationships, although the angular velocity vector or matrix is usually substituted for the time derivative of the orientation matrix.

The EPs are commonly split into a scalar portion $\beta_0$ and a vector portion $\beta$ that has elements $\beta_k, k = 1, 2, 3$. It is implicitly understood that, when relevant, the vector portion $\beta$ and the angular velocity vector $\omega$ are componentized in a body-fixed frame $b^+$ in the following expressions.

**Orientation**

\[
C = 1 - 2\beta_0^2 + 2\beta^T \beta^x
\]

\[
\beta_0^2 = \frac{1}{4} (1 + \varsigma) ; \quad \beta_k^2 = \frac{1}{4} (1 + 2C_{kk} - \varsigma)
\]

\[
\beta_0\beta_1 = (C_{23} - C_{32})/4 ; \quad \beta_0\beta_2 = (C_{31} - C_{13})/4
\]

\[
\beta_0\beta_3 = (C_{12} - C_{21})/4 ; \quad \beta_2\beta_3 = (C_{23} + C_{32})/4
\]

\[
\beta_3\beta_1 = (C_{31} + C_{13})/4 ; \quad \beta_1\beta_2 = (C_{12} + C_{21})/4
\]

Here, $\varsigma$ is the trace of $C$. Equations (11.2) through (11.5) need clarification. These equations constitute Shepperd’s algorithm\(^2\) for computing the EPs from $[C]$. First, the largest $\beta_0^2$ is determined from eq. (11.2). The remaining three parameters are then determined from the appropriate use eq. (11.3)–(11.5).

**Kinematics**

\[
\dot{\beta}_0 = -\frac{1}{2} \beta^T \omega ; \quad \dot{\beta} = \frac{1}{2} (\beta_0 \mathbf{1} + \beta^x) \omega
\]

\[
\omega = -2\beta \dot{\beta}_0 + 2 (\beta_0 \mathbf{1} - \beta^x) \dot{\beta}
\]

---

**Kinetic Energy Expressions.** The kinetic energy function corresponding to different mathematical models can be an important part of forming equations of motion and evaluating work-rate relationships. Consequently, we gather a few expressions on this page.

Point mass model \( T = \frac{1}{2} m v \cdot v \) \hspace{1cm} (12.1)

Rigid body model \( T = \frac{1}{2} m v_c \cdot v_c + \frac{1}{2} \omega \cdot h_c \) \hspace{1cm} (12.2)
\[
= \frac{1}{2} m v_c \cdot v_c + \frac{1}{2} \omega \cdot I_c \omega
\] \hspace{1cm} (12.3)

Flexible beam model \( T = \frac{1}{2} \int_0^\ell \rho \dot{y}(x,t)^2 \, dx \) \hspace{1cm} (12.4)

The flexible beam model considers transverse motion of a beam where the over-dot denotes \( \partial y(x,t)/\partial t \).

It is worth mentioning that the kinetic energy expression for any discrete coordinate system can be written as the sum of three terms labeled \( T_2, T_1, T_0 \).

\[
T = \frac{1}{2} \dot{q}^\top M \dot{q} + \dot{G}^\top \dot{q} + T_0 = T_2 + T_1 + T_0
\] \hspace{1cm} (12.5)

The scalar function \( T_2 \) is quadratic in the generalized velocities, \( T_1 \) is linear in the generalized velocities, and \( T_0 \) is independent of the generalized velocities. When \( T_1 \) and \( T_0 \) are zero, then \( T = T_2 \) only, and we say that the system is a *natural* system.
Lagrange’s Equations. The Lagrangian is a scalar function defined by the difference between the kinetic and potential energies of a system, $L \equiv T - V$. The governing equations of motion can be found from specific derivative operations of $L$. Lagrange’s equations for an $n$ degree of freedom, discrete coordinate system read as follows.

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_k} \right) - \frac{\partial L}{\partial q_k} = Q_k, \quad k = 1, \ldots, n$$  \hspace{1cm} (13.1)

There are a few important facts to note about the Lagrangian and this method of generating equations. First, the function $L$ is written in terms of minimal system coordinates $q$ and their derivatives $\dot{q}$. We say that $L$ is a function of system coordinates and system velocities. Note that the integration of the $\dot{q}$ equations gives the coordinates $q$, and the vector $q$ gives the system configuration. Second, note that $\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right)$ will give rise to system accelerations $\ddot{q}$. So Lagrange’s equations for $n$ independent system coordinates gives rise to $n$ second-order, ordinary differential equations.

A classic example is a point mass in a rotating tube. One end of the tube is fixed, and the tube rotates in a horizontal plane at a known constant rate $\dot{\theta}$. The mass is free to move within the rotating tube and its location is given by the coordinate $r$. The system is the point mass. The inertial velocity vector is $v = \dot{r} \hat{b}_1 + r \dot{\theta} \hat{b}_2$, where the unit vectors $\hat{b}_1$ and $\hat{b}_2$ are part of a reference frame, the $b^+$ frame, that rotates with the tube such that $\hat{b}_3$ is perpendicular to the plane containing the motion. This velocity vector is used to form the kinetic energy function, $T$. There is no potential energy contribution to the Lagrangian function so $L = T$. Moreover, the external forces from contact with the tube are ideal constraint forces so they do not participate in forming the generalized active forces, $Q$. Lagrange’s method gives a single equation of motion for the point mass.

$$m(\ddot{r} - r \dot{\theta}^2) = 0$$  \hspace{1cm} (13.2)
Linear SDOF Models. Many systems can be modeled using one coordinate, and many times a linear differential equation is adequate to model the system response.

\[ m\ddot{x} + c\dot{x} + kx = 0 \quad \text{or} \quad \ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0 \quad (14.1) \]

This is called a linear, single degree-of-freedom model. The phrase single degree-of-freedom is commonly abbreviated as SDOF.

Equation (14.1) contains positive constant coefficients \( m, c, \) and \( k, \) and the right side is zero, which corresponds to an unforced system. The parameter \( m \) is the system mass, \( c \) is the system viscous damping, and \( k \) is the system stiffness. The parameter \( \zeta \) is the system damping factor or damping ratio, \( \zeta \equiv c/(2\sqrt{km}), \) and \( \omega_n \) is the natural frequency, \( \omega_n \equiv \sqrt{k/m}. \)

Solutions to eq. (14.1) constitute the free vibration response of the system. The initial condition, free vibration response for the undamped \( (\zeta = 0) \) and lightly damped cases \( (\zeta < 1) \) are well known.

\[ x(t) = x_0 \cos \omega_n t + \left(\frac{\dot{x}_0}{\omega_n}\right) \sin \omega_n t \quad (14.2) \]

\[ x(t) = \exp(-\zeta\omega_n t) \left( x_0 \cos \omega_d t + \frac{\dot{x}_0 + \zeta\omega_n x_0}{\omega_d} \sin \omega_d t \right) \quad (14.3) \]

Here, \( \omega_d \) is the damped natural frequency defined by \( \omega_d \equiv \omega_n \sqrt{1 - \zeta^2}. \)

There are some instances when the system damping is zero and the “stiffness term” is negative.

\[ m\ddot{x} - kx = 0 \quad (14.4) \]

The response for this system can be written in terms of the hyperbolic sine and cosine functions and the initial condition parameters.

\[ x = x_0 \cosh \gamma t + \left(\frac{\dot{x}_0}{\gamma}\right) \sinh \gamma t \quad (14.5) \]

Here, \( \gamma = \sqrt{k/m} \). The solution reveals that the system response diverges from its initial conditions as time increases. A classic example of this case is the problem of a point mass that travels in a tube that rotates at a known constant rate.
Linear MDOF Models. A extension of SDOF linear systems includes multiple degree-of-freedom (MDOF) linear systems. There are various forms, but the most common is similar to the SDOF case.

\[ M\ddot{x} + C\dot{x} + Kx = f \]  \hspace{1cm} (15.1)

The above matrix equation is written in terms of a mass matrix \( M \), a stiffness matrix \( K \), and a damping matrix \( C \). Usually these matrices are symmetric, but this is not necessarily so. Often times absolute coordinates lead to symmetric system matrices whereas relative coordinates do not. Note that, depending on the system and the linearization, either the stiffness matrix, or damping matrix, or both could be missing from eq. (15.1).

A more general form of linear MDOF system is presented by Hughes.

\[ M\ddot{x} + (C + G)\dot{x} + (K + H)x = f \]  \hspace{1cm} (15.2)

The additional matrices \( G \) and \( H \) are, respectively, gyric and constrained damping matrices. These matrices are skew-symmetric.
Project 1.

1. Show that the time derivative of the kinetic energy for a rigid body is
\[ \dot{T} = f \cdot v_c + \ell_c \cdot \omega. \] The vector \( \ell_c \) consists of pure moments (\( \ell_a \)) and couples (i.e., of the \( r \times f \) variety).

2. Suppose \( f_i \) represents the \( i \)th external force vector acting on a rigid body at a location \( r_i \), and let \( v_i \) be the inertial derivative of \( r_i \). Show that the time derivative of the kinetic energy for a rigid body can be written as
\[ \dot{T} = \sum f_i \cdot v_i + \ell_a \cdot \omega. \]

3. Develop the equations of motion for a tumbling rigid body carrying an internal point mass. Let the rigid body undergo rotational motion only. The internal point mass is constrained to move along an axis that is parallel to a principal axis of the rigid body, and offset from the origin of the rigid body principal frame. Furthermore, the internal point mass is subject to a spring force and viscous damping force along the direction of motion. Let there be no external forces or torques acting on the rigid body.

4. Develop an equation for the kinetic energy, in terms the state variables, for the system in problem 3.

5. Develop an equation for the time derivative of the kinetic energy for the system in problem 3.

6. Consider a set of initial conditions for the system in problem 3. Numerically simulate the motion and plot the kinetic energy versus time. Does the plot ever show a positive slope? Should it?
2 Stability of Motions & an Attitude Regulation Problem
An Introduction. Designing a control law to regulate the attitude of an arbitrary rigid body is a fundamental problem in aerospace engineering. Regulation means bringing a body to rest at a specific attitude configuration. R.E. Mortenson addressed this problem in his 1968 paper, “A globally stable linear attitude regulator,” published in the *International Journal of Control*. He used Lyapunov techniques together with an appropriate set of state variables to show that a linear feedback control law can render a unique equilibrium point to be asymptotically stable for arbitrary initial conditions.

This problem is reviewed here. We begin by discussing the stability of motion around system equilibrium points, and then show how Lyapunov techniques can be used to design feedback control laws that produce a desired outcome.

Notationally, physical vectors, their component matrix representations, and ordinary rectangular matrices that represent a collection of variables, parameters, or constants are written using boldface type with no surrounding brackets. That is, nearly everything becomes, for example, $\mathbf{A}$ or $\mathbf{x}$. 
**Discrete Systems.** Whether Lagrangian techniques or Newton-Euler methods are used, the governing equations of motion for many discrete coordinate systems are written as nonlinear, first-order, ordinary differential equations.

\[ \dot{x} = f(t, x) \]  

(19.1)

The column \( x \) commonly contains the configuration and velocity variables and is traditionally called the state vector. A solution \( x(t) \) is called a state trajectory. Linear systems are a special class of eq. (19.1).

\[ \dot{x} = A(t)x \]  

(19.2)

Systems that are subject to external inputs \( u \) can be put into the form of eq. (19.1) if the controls are taken in feedback form.

\[ \dot{x} = f(t, x, u) \quad \text{with} \quad u = u(t, x) \quad \text{gives} \quad \dot{x} = f(t, x, u(t, x)) \]  

(19.3)

A special class of this type are linear feedback control systems.

\[ \dot{x} = A(t)x + B(t)u \quad \text{becomes} \quad \dot{x} = (A(t) + B(t)K(t))x \]  

(19.4)

The appearance of \( t \) in the right side of eq. (19.1) is taken to mean that time explicitly appears in the governing equations. Such systems are called nonautonomous systems. If \( t \) does not explicitly appear, then a system is called autonomous.

\[ \text{nonautonomous system:} \quad \dot{x} = f(t, x) \]  

(19.5)

\[ \text{autonomous system:} \quad \dot{x} = f(x) \]  

(19.6)

Our primary focus will be on autonomous systems.
**Equilibrium Points.** The equilibrium points of a system are defined as those points in the state space where the system can stay forever. These are identified by setting $\dot{x}$ to zero. (For second-order descriptions, the system velocities and accelerations are set to zero.)

$$\text{equilibrium points: } x_* \text{ such that } f(x_*) = 0 \tag{20.1}$$

Thus, the system trajectory collapses to a single point at an equilibrium point. Equations (20.1) describe a set of nonlinear algebraic equations for the unknowns $x_*$. It is interesting to consider the equilibrium points of a linear autonomous system, $\dot{x} = Ax$.

$$\text{equilibrium points: } x_* \text{ such that } Ax_* = 0 \tag{20.2}$$

There are two specific cases to consider.

1. If $A$ is nonsingular, then $x_* = 0$ is the unique equilibrium point.
2. If $A$ is singular, then $A$ has a null space and any (nonzero) vector that lies in the null space of $A$ defines an equilibrium point.

Consider the second-order system $\ddot{x} + \dot{x} = 0$. The system matrix $A$ for the first-order form reads as follows.

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \tag{20.3}$$

This matrix is singular with null space $N = [c, 0]^T$, where $c$ is an arbitrary constant. Therefore, this system has infinitely many equilibrium points.

A linear autonomous system has either a unique equilibrium point or infinitely many. There is nothing in between.
A nonlinear autonomous system can have one, or several, or infinitely many equilibrium points. Consider a nonlinear second-order system.

$$\ddot{x} + x - x^3 = 0$$  \hspace{1cm} (21.1)

The equilibrium points correspond to $\dot{x} = 0$ together with $x = 0$ or $x = \pm 1$. Thus, there are three equilibrium points. As another example, consider a simple pendulum model.

$$\ddot{x} + \sin x = 0$$  \hspace{1cm} (21.2)

The equilibrium points correspond to $\dot{x} = 0$ together with $x = 0, \pm \pi, \pm 2\pi, \ldots$ Thus, there are infinitely many (mathematical) equilibrium points that correspond to the two physical (equilibrium) points (hanging straight down or inverted straight up).

An equilibrium point of a nonlinear system can always be taken to be $x_\ast = 0$. This is done with a simple change of coordinates. To illustrate, suppose $\dot{x} = f(x)$ with $f(x_\ast) = 0$ and $x_\ast \neq 0$. Consider a linear change of coordinates $z = x - x_\ast$, which gives $\dot{z} = \dot{x}$. Then $\dot{z} = f(x) = f(z + x_\ast)$. An equilibrium point for the new coordinates is defined by $z_\ast$ such that $f(z_\ast + x_\ast) = 0$. But because $f(x_\ast) = 0$, then $z_\ast = 0$.

Consequently, without the loss of generality, we can always take an equilibrium point to be zero.
**Motion Near a Trajectory.** Sometimes we are interested in behavior of a system around an equilibrium point, and other times we are interested in behavior around a trajectory. The motion of a nonlinear autonomous system around an equilibrium point is governed by a nonlinear autonomous system. But the motion of a nonlinear autonomous system around a trajectory is governed by a nonlinear, nonautonomous system of equations.

To illustrate the latter, consider a system $\dot{x} = f(x)$. The state trajectory associated with initial condition $x(0) = x_0$ will be labeled $x_*(t)$. The state trajectory that emanates from a nearby initial condition, $x_0 + \delta$, will be labeled $x(t)$. We wish to develop equations that govern the nearby motion around $x_*(t)$.

An error vector can be formed by differencing the state trajectories, $e(t) = x(t) - x_*(t)$, and this definition leads to an error dynamics equation.

$$\dot{e} = f(x_* + e) - f(x_*) \quad (22.1)$$

But because $x_*(t)$ is a known trajectory, i.e., $x_*(t)$ is a known function of time, the error dynamics are a function of the error state and time.

$$\dot{e} = g(t, e) \quad \text{with} \quad e(0) = \delta \quad (22.2)$$

The error dynamics are governed by a nonlinear, nonautonomous system of equations.

As an example, consider a nonlinear oscillator.

$$\ddot{x} + x + x^3 = 0 \quad (22.3)$$

Motions near a known, nominal trajectory $x_*(t)$ of this system are governed by the following nonautonomous equation.

$$\ddot{e} + e + e^3 + 3e \dot{x}_* (e + x_*) = 0 \quad (22.4)$$
Stability Definitions. Loosely speaking, stability has to do with the well-behavedness of system motion around a desired operating point or trajectory. Four ways to describe local stability are stable, unstable, asymptotically stable, and exponentially stable.

*Stable*: Motion near the equilibrium point \( x_* = 0 \) is stable if \(^3\) for any positive constant \( R > 0 \) there exists a positive constant \( r > 0 \) such that \( \|x(0)\| < r \) implies \( \|x(t)\| < R \) for all \( t > 0 \). This says that trajectories starting close to \( x_* \) will remain close to \( x_* \). This type of behavior is also called stable in the sense of Lyapunov or Lyapunov stable or marginally stable.

*Unstable*: Motion near the equilibrium point \( x_* = 0 \) is unstable if it is not stable. This does not necessarily mean that a state trajectory blows up.

*Asymptotically Stable*: Motion near the equilibrium point \( x_* = 0 \) is asymptotically stable if it is stable and there is some positive constant \( c > 0 \) such that \( \|x(0)\| < c \) implies \( x(t) \to 0 \) as \( t \to \infty \). This says that trajectories not only remain close to \( x_* \), but states actually converge to zero as time goes on.

*Exponentially Stable*: Motion near the equilibrium point \( x_* = 0 \) is exponentially stable if there exists two strictly positive numbers \( \alpha \) and \( \beta \) such that \( \|x(t)\| \cdot \alpha \|x(0)\| \exp(-\beta t) \) for all \( t > 0 \) within some region \( r \) around \( x_* \). This says that trajectories go to zero faster than some exponential function. This type of stability provides a measure of performance and is like a nonlinear version of a time constant for a linear system.

*Global*: The above definitions characterize motion near an equilibrium point. If asymptotic or exponential stability holds for any initial condition, then \( x_* \) is asymptotically or exponentially stable in the large, or globally asymptotically stable or globally exponentially stable.

\(^3\)Or more simply stated, the equilibrium point \( x_* \) is stable if ...
**Stability of Linear Systems.** An equilibrium point of a linear autonomous system (or a linear time-invariant system) is either (marginally) stable or asymptotically stable or unstable. This is seen from the superposition principle.

Asymptotically stability of an equilibrium point is always global and exponential, whereas instability always implies that the state trajectories blow-up. The refined notions of stability, like local versus global, or stable versus asymptotically stable versus exponentially stable, are not needed when performing linear systems analysis.

Consider the trivial example $\dot{x} = ax$. An equilibrium point of this system is $x_0 = 0$.

1. If $a = 0$, then motion near the equilibrium point beginning at $x_0$ is marginally stable. Indeed, the state trajectory is a constant corresponding to the initial condition.

2. If $a < 0$, then motion near the equilibrium point beginning at $x_0$ is asymptotically stable, and the state trajectory is $x(t) = x_0 \exp(at)$. Thus, the equilibrium point is exponentially stable from any initial condition.

3. If $a > 0$, then motion near the equilibrium point beginning at $x_0$ is unstable. The state trajectory is $x(t) = x_0 \exp(at)$, which reveals that the state trajectory blows-up with respect to time for all initial conditions.
Lyapunov’s Linearization Method. Lyapunov devised two methods to test the stability of motion near an equilibrium point. The one discussed here is based on a linear approximation of the autonomous nonlinear system.

The method begins with a Taylor series expansion of the nonlinear vector function \( f(x) \) around the equilibrium point \( x_* = 0 \) for the system \( \dot{x} = f(x) \).

\[
f(x) = f(x_* + \delta) \approx f(x_*) + (\frac{\partial f}{\partial x})_{x_*} \delta = (\frac{\partial f}{\partial x})_{x_*} \delta \quad (25.1)
\]

But \( \delta \) is simply \( x \) because \( x_* \) is taken as zero, and a matrix \( A \) can be introduced to rename the Jacobian matrix of partials, \( A \equiv (\frac{\partial f}{\partial x})_{x_*} \). Thus, a linearized approximation of the nonlinear equations of motion can be formed.

Actual nonlinear system: \( \dot{x} = f(x) \quad (25.2) \)

Linearized system: \( \dot{x} = Ax \quad (25.3) \)

Lyapunov proved the following:

1. If the linearized system is strictly stable, which means all eigenvalues of \( A \) are strictly in the left-half plane, then motion of the actual nonlinear system near the equilibrium point \( x_* \) is asymptotically stable.

2. If the linearized system is unstable, which means at least one eigenvalue of \( A \) is strictly in the right-half plane, then motion of the actual nonlinear system near the equilibrium point \( x_* \) is unstable.

3. If the linearized system is marginally stable, which means all eigenvalues of \( A \) are in the left-half plane but at least one is on the imaginary axis, then the linearized system is unrevealing. The motion of the actual nonlinear system near the equilibrium point \( x_* \) may be stable, asymptotically stable, or unstable.