Periodicity Characterization of Orbital Prediction Error and Poisson Series Fitting

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Abstract

Orbital prediction error covariance is important for conjunction risk assessment of space objects. Publicly available NORAD Two-Line Element Sets (TLE) contain no associated error or accuracy information, the historical data-based method is one choice for those objects for which only TLE data are available. Most of current TLE error analysis methods use polynomial fitting, which can not represent the periodic characteristic of the prediction error within one orbital period. To solve this problem, this paper conducts a pair-wise comparison of the TLE predicted state within one period interval centered in the epoch of TLEs. An error fitting method taking account of the periodic characteristic is presented, Poisson series are chosen as an error fitting function describing the variation of the standard deviation with respect to predicting the time and phase of the objects in their orbit. In the processing of conjunction risk assessment, the epoch time difference and the mean anomalies of the two objects can be provided by the closest approach analysis. Using the time difference and mean anomaly in the Poisson series one can obtain the positional error information of the space object at the time of closest approach, which is important in the calculation of the collision probability.

Keywords: Two Line Element Sets (TLE); orbit error; Poisson series; periodic characterization; collision risk assessment

1. Introduction

With the continuous increase of on-orbit space objects, conjunction assessment and collision avoidance are increasingly important, for the risk of collisions between spacecraft and space objects increases rapidly. The U.S. and Russian satellites’ collision event, that occurred on 10 Feb 2009, has highlighted the practical necessity of conjunction assessment and collision avoidance much more. Currently, the main criterion in collision risk assessment is the collision probability (Pc) between two objects. The Pc is based not only on the closest approach distance

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at the time of closest approach (TCA), but also on the conjunction geometry and the uncertainties (or covariance) associated with the space objects’ state vectors.

The calculation of Pc needs information of the position and velocity vectors and associated error covariance of the two objects at the TCA. How to obtain the orbital error covariance information of space objects is a key problem in the conjunction risk analysis. For most of the non-cooperative objects (such as satellites not owned/operated by us, rockets’ bodies, and space debris) the orbital data used in conjunction analysis are the North American Aerospace Defense Command (NORAD)’s Two-Line Elements Sets. NORAD uses the tracking data from the Space Surveillance Net (SSN) to maintain a space catalog, which includes both operational and dead satellites, launch vehicles and rocket bodies, and space debris. The format in which orbital elements are released is called the Two-Line Element Set, or TLE. TLEs are used with the Simplified General Perturbation 4 (SGP4) analytical orbital model to determine an object’s position and velocity vectors at a specific time. However, no associated accuracy level, or uncertainty, is provided with TLEs. Currently, NORAD TLEs released for public use contain no associated accuracy information. Therefore, an accurate and practical TLE error analysis and assessment method is needed for conjunction assessment and collision avoidance.

For a non-cooperative object only having TLE data, the historical data-based error statistical analysis method is a feasible choice. Currently, there are two classes of methods included: 1) initial error analysis and 2) error evolution analysis.

Initial error analysis based on historical TLE data obtains the initial covariance information at the epoch of the TLE. Klinkrad, Krag, and Flohrer from ESA introduced pre-defined look-up tables for the initial covariance that are sorted by eccentricity, perigee altitude and inclination. This information was applied to the propagation of the state and covariance. The covariance look-up tables were generated by comparing states derived directly from the TLE data to states resulting from an orbit determination using pseudo-observations followed by a numerical propagation. Those pseudo-observations were derived from the TLE data, the desired covariance information was obtained from the statistical analysis of the residuals in the radial, along-track, out-of-plane coordinate.

Initial error analysis can only obtain the covariance information at the epoch of TLE. To obtain covariance information at the time of closest approach (TCA) one should propagate the state and covariance forward by a numerical model, which is complex and computing-inefficient. Differing from initial error analysis, error evolution analysis can obtain evolution characteristics of the prediction error with respect to the time and other parameters directly. The basic method is comparing states at, or around, the TLE’s epoch to the predicted states at the epoch using prior TLEs and the SGP4 propagation model. The residuals are statistically analyzed to get error information and the temporal evolution of statistical quantities such as mean and covariance.

Peterson [4] from Aerospace Corporation, Deguine[5] from CNES, and Osweiler[6] from US Air University have studied the error evolution analysis method based on historical data. In their study general error information was generated by relying solely upon the publicly available TLEs. This method operated on a time series of TLEs for a given satellite object, started with a pair-wise differencing of a set of TLEs for a certain time span, then computed residuals in a satellite-based coordinate system, performed a quadratic least squares fit, calculated a standard deviation from those, and removed outliers. They then calculated a second best fit of the remaining residuals, and used that as the time-varying error estimation.

Current TLE error evolution analysis compare predicted states at the TLE’s epoch, the error fitting function is a temporal polynomial, and the TLE prediction error’s periodicity in an orbital
period is not taken into account. The TLE’s prediction errors are different, not only at a different time, but also at different positions on the orbit, especially the highly eccentric orbit (HEO). For the highly eccentric orbit, the perturbation situations are varied from the orbit’s perigee to apogee remarkably, the error will be large near perigee and smaller near apogee, causing the uncertainty ellipsoid to breathe as the object moves on its orbit. Therefore, for HEO objects periodicity characterization of the TLE prediction error must be done to get more accurate covariance information. Most objects’ TLE are released at the epoch, when the objects are passing their orbit’s ascending node when the argument of latitude is about zero. So the residuals at the epoch can not reflect the periodic characteristics of the TLE error.

To solve this problem, this paper assumes that the TLE’s prediction errors in an orbital period centered on its epoch are small and that the states can be used as a reference. It compares predicted states in the whole period, so that the residuals can reflect the periodicity of the TLE error. Poisson series, rather than polynomial series, are taken as error fitting functions describing the variation of the standard deviation with respect to predicting times and phases of objects in their orbit. The error analysis and fitting method presented in this paper can not only be used to handle TLE sets, but also can be used to handle any other sequential analytical or numerical orbital elements, because the process did not require how the state vector was generated. Therefore, the method can be extended to other orbital models and elements.

2. Methodology

2.1 Selection of space objects

To begin with, we select four satellites as examples to demonstrate and validate the error analysis and fitting method. Each of the four objects belongs to an individual orbital type: low Earth orbit (LEO), medium Earth orbit (MEO), highly eccentric orbit (HEO), and geostationary orbit (GEO). The selected satellites and some of their orbital parameters are summarized in Table 1. The number of historical TLE sets of each of the objects are also included in the table, all TLE used in this study are obtained from www.space-track.org. For the purpose of easy reference, in the following text each object in Table 1 is identified by its orbital type, for example, we will refer to Cosmos-2251 as LEO object and to SLOSHSAT as HEO object, and so on.

<table>
<thead>
<tr>
<th>Orbit type</th>
<th>Catalog number</th>
<th>Satellite Name</th>
<th>Inclination /deg</th>
<th>Perigee altitude /km</th>
<th>Apogee altitude /km</th>
<th>TLE Sets</th>
</tr>
</thead>
<tbody>
<tr>
<td>LEO 1</td>
<td>22675</td>
<td>Cosmos-2251</td>
<td>74.04</td>
<td>776</td>
<td>799</td>
<td>612</td>
</tr>
<tr>
<td>MEO 1</td>
<td>32711</td>
<td>Navstar-62</td>
<td>55.55</td>
<td>20096</td>
<td>20267</td>
<td>679</td>
</tr>
<tr>
<td>HEO 1</td>
<td>28544</td>
<td>SLOSHSAT</td>
<td>7.08</td>
<td>245</td>
<td>35108</td>
<td>508</td>
</tr>
<tr>
<td>GEO 2</td>
<td>25010</td>
<td>TelStar-10</td>
<td>0.06</td>
<td>35771</td>
<td>35802</td>
<td>821</td>
</tr>
</tbody>
</table>
2.2 Generation of residual data

To generate the residual data of the TLE’s prediction, the straightforward approach is calculating the difference between the predicted state vector (position and velocity) and the “real” state vector of the space object at specific times.

\[ \Delta X = X_{\text{pred}} - X_{\text{real}} \]  

However, in practice the “real” state vector \( X_{\text{real}} \) is unknown, which must be approximated. The only available information in the historical data-based error analysis is the TLE data itself, so some kind of predicted value must be used to approximate the ‘real’ value. Generally, the prediction error at TLE’s epoch is small (but not always the minimum), so in the prior works the state vectors derived from each TLEs at their epoch are taken as an approximation of the “real” states. Most objects’ TLE are released when objects are passing their orbit’s ascending node, where the argument of latitude is about zero, so the residuals at epoch can not reflect the periodic characteristics of the TLE error.

This paper assumes that the TLE’s prediction errors in a orbital period centered on its epoch are small and that the states can be used as a reference. The prediction state vectors of prior TLEs to this time span are used as a prediction. Differencing the predictions and reference, one can obtain residual data. Take an orbital period-long time span centered on the \( j \) th TLE’s epoch \( t_j \), as seen in Fig. 1. This time span is then separated into \( n \) segments so we have \( n+1 \) time points.

\[ t_{j,k} = t_j - \frac{T}{2} + k \frac{T}{n}, \quad k = 0, 1, 2, \ldots n \]  

Fig. 1 illustrates a case where \( n=10 \). At each of the time points \( t_{j,k} \) in the \( j \) th time span a predicted state vector \( X(t_i / t_{j,k}) \) is derived from the \( i \) th (\( i < j \)) TLE set, and the state vector \( X(t_{j,k}) \), derived from the \( j \) th TLE set itself, can be obtained, respectively. Comparing \( X(t_i / t_{j,k}) \) and \( X(t_{j,k}) \) we can get the residual vector \( \Delta X(t_i / t_{j,k}) \), the corresponding temporal difference \( \Delta t_{i,j,k} \), and the mean anomaly \( M_{j,k} \) at time points \( t_{j,k} \).

![Fig. 1. Generation of difference between the predicted and reference state data in the whole period.](image)

To get the \( i \) th TLE’s prediction residual data, the state vector is propagated to every following TLE’s orbital period-long time span, and compared with the state vector derived from following TLE to get the residual vector. Repeat the process until the temporal difference is equal or greater than the time threshold, \( \Delta t_{i,j} > t_T \). So the \( i \) th TLE’s prediction residual data and
the corresponding temporal differences and mean anomalies can be obtained. All residual vectors are associated with a time parameter and an angular parameter.

\[
\begin{bmatrix}
\Delta t_{i,1} & \Delta U_{i,1} & \Delta N_{i,1} & \Delta W_{i,1} & \Delta V_{U1} & \Delta V_{N1} & \Delta V_{W1} \\
\Delta t_{i,2} & \Delta U_{i,2} & \Delta N_{i,2} & \Delta W_{i,2} & \Delta V_{U2} & \Delta V_{N2} & \Delta V_{W2} \\
\Delta t_{i,3} & \Delta U_{i,3} & \Delta N_{i,3} & \Delta W_{i,3} & \Delta V_{U3} & \Delta V_{N3} & \Delta V_{W3} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\Delta t_{i,N} & \Delta U_{i,N} & \Delta N_{i,N} & \Delta W_{i,N} & \Delta V_{U,N} & \Delta V_{N,N} & \Delta V_{W,N}
\end{bmatrix}
\]

Fig. 2. Generation of difference between predicted and reference state data.

Repeating the process from first to the N_{th} TLE set as illustrated in Fig. 2, we can obtain N residual vectors and the corresponding temporal differences and mean anomalies.

\[
\{\Delta X_1 (\Delta t_1, M_1), \Delta X_2 (\Delta t_2, M_2), \cdots, \Delta X_N (\Delta t_N, M_N)\}
\]

(3)

State residual vectors are transformed from the Earth Centered Inertial (ECI) coordinate system to the UNW (in-track, normal, and cross-track) satellite-based coordinate system, as

\[
\Delta X = [\Delta U, \Delta N, \Delta W, \Delta V_U, \Delta V_N, \Delta V_W]
\]

(4)

Therefore, a N×8 residual data matrix is established, the first column of the matrix is the temporal difference \(\Delta t\), the second column is the mean anomaly, and the third to eighth columns are the six components of the residual state vector in the UNW coordinate system.

So far, we’ve gotten a sample TLE prediction residual data matrix, this data matrix is the basis for the error analysis, all the analysis below is based on this data matrix.

Let the temporal difference threshold \(t_t\) = 3 days and the segment number in an orbital period \(n\) = 20. Fig. a1 and Fig. a2 in the Appendix illustrate the TLE’s prediction residual state components in the UNW coordinate system vs. the temporal difference and vs. the mean anomaly of HEO1, respectively. It’s evident that there are periodic characteristics of the state vector errors in all six components. Fig. a1 and Fig. a2 show that the prediction errors near the perigee, where mean anomalies are about zero, are larger than near the apogee, where mean anomalies are about ±180°. There is only one time parameter in the polynomial fitting function that can not reflect the variation of error in an orbital period.

2.3 Residual data preprocessing
Before statistical analysis and fitting of the TLE’s error, it’s necessary to preprocess the residual data. The preprocessing of residual data consists of data binning and outlier detection.

In Reference [6] the residual data were binned according to the corresponding epoch time difference $\Delta t$. Without considering the regular and discrete characteristics of the TLE’s epoch, reference [6] supposed the epoch time difference resulting from the pair-wise comparison can be any value, so it binned the data by the ratio of the epoch time difference to one day, and bin intervals of one day were chosen.

The residual data in this paper have two parameters, Fig. 3(a) illustrates the temporal difference and mean anomaly of the residual data of a LEO object, so two-dimensional data binning must be done. Considering the regular and discrete characteristics of TLE’s epoch, the time parameters are binned according to the ratio of temporal difference $\Delta t$ to one orbital period $T$. The ratios of temporal difference $\Delta t$ to current orbital period $T$ are calculated, when we get the temporal difference, $n_{\Delta t/T} = \Delta t / T$. Generally, the ratios $n_{\Delta t/T}$ are not integers, so we round them to the nearest integer $N_{\Delta t/T}$. Then, the time parameters can be binned according to number $N_{\Delta t/T}$.

As described before, an orbital period is separated into $n$ segments, so each segments’ length is $M_{\text{bin}} = 360/n$. Similarly, the ratios of the mean anomaly $M$ to $M_{\text{bin}}$ are calculated when we get the mean anomaly, $n_M = M / M_{\text{bin}}$. Generally, the ratios $n_M$ are not integers; they are rounded to the nearest integer $N_M$. Then, the angular parameters can be binned according to number $N_M$. Thus, for all residual data two integral parameters are obtained, residual data can be binned two-dimensionally according to the two parameters. Fig. 3(b) illustrates every data bins’ time and angular parameters. The threshold of prediction time is $t_\gamma$, the orbital period is $T$, the minimum value of $N_{\Delta t/T}$ is 0, the maximum value of $N_{\Delta t/T}$ is $\text{int}(t_\gamma / T)$, the minimum value of $N_M$ is 0, and maximum value of $N_M$ is $n$. The total number of data bins will be

$$N_{\text{bin}} = \left(\text{int}\left(t_\gamma / T\right) + 1\right)\left(n_{\text{period}} + 1\right)$$  (6)

Fig. 3. Residual data two-dimensional binning. (a: left, before binning; b: right, after binning)
Due to an orbit maneuver, TLE generating error, or other reasons, it’s inevitable that there are outliers in the TLE sets; consequently there are outliers in the residual data, too. After data binning, outliers in each data bin must be detected. A Mahalanobis distance-based outlier detecting method is used. If the Mahalanobis distance $d_M$ from a data point to the distributing center of the bin exceeds a distance threshold (such as 3 or 5), then this data point will be seen as an outlier and eliminated.

![Fig. 4. In-track (U) position error standard deviations for each data bin of four objects.](image)

We calculate mean temporal difference $t_k$, means of mean anomaly $M_k$, and error standard deviations in three directions $S_u$, $S_n$, $S_w$ of each data bin. Fig. 4 shows the in-track position error standard deviations for each data bin of four objects. From Fig. 4 we also can see the temporal evolution and period characteristics of the TLE’s prediction error.

2.4 Poisson series and coefficient fitting

Poisson series appear very often in celestial mechanics and more generally in perturbation theories for non-linear mechanics or non-linear differential equations. Poisson series, also called polynomial-trigonometric series, have the general form as Eq. (7).

$$P = \sum_{h_{k_1}, \ldots, h_{k_n}} \sum_{j_{k_1}, \ldots, j_{k_n}} C_{h_{k_1}, \ldots, h_{k_n}}^{j_{k_1}, \ldots, j_{k_n}} x_1^{j_{k_1}} x_2^{j_{k_2}} \cdots x_n^{j_{k_n}} \left( \sin \frac{\phi}{\cos} \right) (j_{k_1} \phi + \cdots + j_{k_n} \phi_m)$$

(7)
Here \( x_1, x_2, \ldots, x_n \) are polynomial variables, and \( \phi_1, \phi_2, \ldots, \phi_m \) are trigonometric variables, respectively. Coefficients \( C_i^j \) may be represented as rational, floating-point, or complex numbers. The summation is performed over all integer values of indices \( i \) to \( i_n \) and \( j \) to \( j_m \). Besides the fact that they arise frequently in Celestial Mechanics, Poisson series have several properties of importance for symbolic manipulation. The results of addition, subtraction, multiplication and differentiation (with respect to any of the variables) of Poisson series are again Poisson series; if the polynomial arguments are raised to non-negative powers, this also holds for the result of substituting one Poisson series for one of the polynomial arguments of another, and for the case of integration with respect to a polynomial argument. Integration with respect to a trigonometric argument also results in a Poisson series, if the original series contains no terms which are constant with respect to that argument (i.e., the 'periodic part' of the series). Finally, a Poisson series is obtained, if a substitution of one series for an angular variable in another series is made by means of an expansion in (truncated) power series.

There is only one polynomial variable (temporal difference) and one trigonometric variable (mean anomaly) in the TLE prediction residual data, hence, Eq. (7) could be simplified to (1,1) style Poisson series

\[
P(t, \phi) = \sum_{i=0}^{n} \sum_{j=0}^{m} t^i \left( A_{i,j} \cos j\phi + B_{i,j} \sin j\phi \right)
\]

(8)

where \( t \) and \( \phi \) are polynomial variables and trigonometric variables, \( A_{i,j} \) and \( B_{i,j} \) are Poisson coefficients, \( n \) and \( m \) are the maximum orders of the polynomial terms and trigonometric terms, respectively. Eq. (8) can be expanded to

\[
P(t, \phi) = A_{0,0} + A_{1,0} \cos \phi + A_{2,0} \cos 2\phi + \cdots + A_{n,0} \cos n\phi + B_{0,1} \sin \phi + B_{0,2} \sin 2\phi + \cdots + B_{0,m} \sin m\phi + A_{1,1} t \cos \phi + A_{1,2} t \cos 2\phi + \cdots + A_{1,m} t \cos m\phi + B_{1,1} t \sin \phi + B_{1,2} t \sin 2\phi + \cdots + B_{1,m} t \sin m\phi + A_{2,1} t^2 \cos \phi + A_{2,2} t^2 \cos 2\phi + \cdots + A_{2,m} t^2 \cos m\phi + B_{2,1} t^2 \sin \phi + B_{2,2} t^2 \sin 2\phi + \cdots + B_{2,m} t^2 \sin m\phi + \cdots + A_{n,0} t^n \cos \phi + A_{n,1} t^n \cos 2\phi + \cdots + A_{n,m} t^n \cos m\phi + B_{n,1} t^n \sin \phi + B_{n,2} t^n \sin 2\phi + \cdots + B_{n,m} t^n \sin m\phi
\]

(9)

Eq. (9) indicates that the expanded form of (1,1) style Poisson series with one polynomial variable and one trigonometric variable consists of three parts:

1. Polynomial terms, the corresponding coefficients are \( A_{i,0} (i = 0, \ldots, n) \), the blue terms in Eq. (9);
2. Trigonometric terms, the corresponding coefficients are \( A_{0,j} (j = 1, \ldots, m) \) and \( B_{0,j} (j = 1, \ldots, m) \), the red terms in Eq. (9);
3. Mixed terms, the corresponding coefficients are \( A_{i,j} (i = 1, \ldots, n, j = 1, \ldots, m) \) and \( B_{i,j} (i = 1, \ldots, n, j = 1, \ldots, m) \), the black terms in Eq. (9);

Because \( \sin j\phi = 0 \) when \( j = 0 \), coefficients \( B_{i,0} (i = 0, \ldots, n) \) are meaningless, the number of coefficients is \((n+1)(2m+1)\). Eq. (9) can also be written as vector and matrix form
\[ P(t, \phi) = \begin{bmatrix} 1 & t & t^2 & \cdots & t^n \end{bmatrix} \begin{bmatrix} A_{0,0} & A_{0,1} & \cdots & A_{0,m} \\ A_{1,0} & A_{1,1} & \cdots & A_{1,m} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n,0} & A_{n,1} & \cdots & A_{n,m} \end{bmatrix} \begin{bmatrix} 1 \\ \cos \phi \\ \vdots \\ \cos m \phi \end{bmatrix} + \begin{bmatrix} B_{0,0} & B_{0,1} & \cdots & B_{0,m} \\ B_{1,0} & B_{1,1} & \cdots & B_{1,m} \\ \vdots & \vdots & \ddots & \vdots \\ B_{n,0} & B_{n,1} & \cdots & B_{n,m} \end{bmatrix} \begin{bmatrix} 0 \\ \sin \phi \\ \vdots \\ \sin m \phi \end{bmatrix} \]

(10)

Let the variable vectors be

\[ X(t) = \begin{bmatrix} 1 & t & t^2 & \cdots & t^n \end{bmatrix}^T \]
\[ Y(\phi) = \begin{bmatrix} 1 & \cos \phi & \cos 2\phi & \cdots & \cos m\phi \end{bmatrix}^T \]
\[ Z(\phi) = \begin{bmatrix} 0 & \sin \phi & \sin 2\phi & \cdots & \sin m\phi \end{bmatrix}^T \]

(11)

Let the coefficient matrices be

\[ A = \begin{bmatrix} A_{0,0} & A_{0,1} & \cdots & A_{0,m} \\ A_{1,0} & A_{1,1} & \cdots & A_{1,m} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n,0} & A_{n,1} & \cdots & A_{n,m} \end{bmatrix}, B = \begin{bmatrix} 0 & B_{0,1} & \cdots & B_{0,m} \\ 0 & B_{1,1} & \cdots & B_{1,m} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & B_{n,1} & \cdots & B_{n,m} \end{bmatrix} \]

(12)

So the vector and matrix form of the (1,1) style Poisson is

\[ P(t, \phi) = X^T(t) \left[ AY(\phi) + BZ(\phi) \right] \]

(13)

To describe the temporal evolution and periodic characteristics of the TLE’s prediction error by Poisson series, we fit the Poisson series over the full available data bins using the least square method. For a certain direction, we have a discrete data set \( \{ t_k, \phi_k, p_k \}_{k=1}^N \), the Least Square method finds the Poisson coefficient matrix (PCM) \( A \) and \( B \), and makes the function Eq. (14) to be minimum.

\[ F = \sum_{k=1}^N \left[ p_k - p(t_k, \phi_k) \right]^2 \]

(14)

The Least square method requires that the number of coefficients to be fitted is less than the number of data sets. That means in this work that the following inequality must be satisfied.

\[ (n+1)(2m+1) < N_{\text{bin}} \]

(15)

where \( n \) and \( m \) are the maximum orders of the polynomial terms and trigonometric terms, and \( N_{\text{bin}} \) is the number of data bins. Substituting Eq. (6) into Eq. (15), we can obtain

\[ (n+1)(2m+1) < \left( \left\lceil \frac{t_T}{T} \right\rceil + 1 \right)(n+1) \]

(16)

From Eq. (16) we know, if \( n \) and \( m \) satisfy the following inequalities, the Poisson coefficient matrix could be fitted using the least square method.

\[ n < \left\lceil \frac{t_T}{T} \right\rceil, \quad m < \frac{n}{2} \]

(17)
3. Results and analysis

With the methodology developed, the residual data can be generated and preprocessed, the Poisson coefficient matrices of each direction can be fitted using the least square method. Fig. 5 illustrates the in-track error standard deviation’s Poisson series fitting curved surfaces of four objects.

![Fig. 5.](image)

**Fig. 5.** In-track error standard deviation’s Poisson series fitting curved surfaces of four objects.

Every part of the Poisson coefficient matrix $A$ and $B$ have specific significance. The first column of matrix $A$ are polynomial coefficients, describing the secular evolution of the prediction error. The first column of matrix $B$ is defined to be zeros. The first row, except the first element in both $A$ and $B$, are trigonometric coefficients, describing the periodic characteristics of the prediction error. Other elements in $A$ and $B$ are mixed coefficients, describing the periodic term whose amplitude grows with time.

3.1 Effect of polynomial term

The first column of the Poisson coefficient matrix $A$ is the polynomial coefficient of the Poisson series, describing the secular evolution of the prediction error. If we only take this part of $A$ and set the other elements of $A$ and each element of $B$ to be zero, the error standard deviation curves derived from the Poisson series of an HEO object are shown in Fig. 6. The normal (N) and cross-track (W) errors are relatively small in magnitude and have near-constant growth rates, the in-track (U) errors, however, are much larger and grow more rapidly. The in-
track (U) component’s standard deviation is one order of magnitude larger than the other two components.

![Graph showing error standard deviation curves](image)

Fig. 6. Error standard deviation curves derived from Poisson series only consist of polynomial coefficients.

3.2 Effect of trigonometric term

The elements in the first row, except the first one in both $A$ and $B$, are trigonometric coefficients of the Poisson series, describing the periodic characteristics whose amplitude keep constant. Set all the elements in $A$ and $B$, except their first row and first column, to be zero, Fig. 7 illustrates the in-track (U) error standard deviation of the HEO object with respect to time and mean anomaly. Comparing Fig. 7 to Fig. 6 we know that the trigonometric term of the Poisson series adds periodic variation to the polynomial error evolution function.

![Graph showing error standard deviation curve surface](image)

Fig. 7. The error standard deviation curve surface derived from the Poisson series consists of both polynomial and trigonometric coefficients.

3.3 Effect of mixed term
The elements in $A$ and $B$, except their first row and first column, are mixed term coefficients, describing the periodic term whose amplitude grows with time. The actual characteristic of the TLE’s prediction error is a combination of polynomial effect, trigonometric effect, and mixed effect, as illustrated in Fig. 5. Polynomial, trigonometric, and mixed terms of Poisson series are combined together, describing the variation of the prediction error with respect to the temporal difference and location in an orbital period.

In order to illuminate the significance of taking the mean anomaly as a variable and using the Poisson series as the fitting function, let the temporal difference $\Delta t=5$ days, Fig. 8 illustrates the in-track, normal, and cross-track error standard deviations curves of the HEO object with respect to the mean anomaly.

From Fig. 8 we know that: (1) the orbital error’s in-track component is one or two orders of magnitude larger than the normal and cross-track components. (2) All in-track, normal, and cross-track errors have periodic characteristics, but the variation of the in-track error in an orbital period is more notable. (3) In-track error is maximum near the orbital perigee ($M=0$) and is small near the apogee ($M=\pm 180^\circ$), whereas the normal and cross-track errors are maximum when $M=\pm 90^\circ$ and are small both near the perigee and apogee. In conclusion, it’s significant to take periodic characteristics into account and introduce the angular variable to the evaluation of the TLE’s error.

4. Conclusion

Based on historical NORAD TLE data, this paper has presented the methodology for periodicity characterization and Poisson series fitting of orbital prediction error. Most of current TLE error analysis methods use polynomial fitting, which can not represent the periodic characteristic of the prediction error within one orbital period. This paper conducted the pairwise comparison of the TLE predicted state within one period interval centered on the epoch of the TLEs. Poisson series were chosen as the error fitting function describing the variation of the standard deviation with respect to the predicting time and phase of the objects in their orbit. Through the methodology presented in this paper we can obtain Poisson coefficient matrices of the error standard deviation. In the processing of the conjunction risk assessment, the epoch time difference and the mean anomalies of the two objects can be provided by the closest approach.
analysis. Substituting time difference and mean anomaly into the Poisson series, one can obtain the positional error information of the space object at the time of closest approach, which is important in the calculation of collision probability.

It is necessary to point out that the error analysis and fitting method presented in this paper can not only be used to handle TLE data, but also to handle any sequential osculating or mean element sets with an analytical or numerical orbital model, because the process did not depend on how the state vector was generated. Therefore, the method can be extended to other orbital models and elements.

Improvement: This work may be improved in two aspects. First, it is too rough to take predicted states at or near the epoch to be references. TLE sets are generated by trajectory sampling and differential correction, it is the best fit to the observations within the fitting interval. This does not imply that the predicted states at epoch are accurate, and the errors at epoch are not even necessarily the minimum. One alternative method is taking states resulting from an orbit determination using pseudo-observations derived from the TLE data followed by a numerical propagation as the reference states. This study is under progress now.

Limitation: In this method each TLE is propagated forward to the intervals centered on epochs of subsequent ones and the states at those intervals are compared. This method tests the consistency between subsequent TLEs. Even if we assume unbiased measurements and error-free orbit determination and TLE generation, successive TLE’s might be correlated due to model errors. Such errors cannot be detected with this method. This is also the limitation of an error analysis only based on historical orbit data.

References


Appendix

Fig. a1. TLE’s prediction residual state components in UNW coordinate system vs. temporal difference of LEO object.

Fig. a2. TLE’s prediction residual state components in UNW coordinate system vs. mean anomaly of LEO object.
Fig. a3. TLE’s prediction residual state components in UNW coordinate system vs. temporal difference of MEO object.
Fig. a4. TLE’s prediction residual state components in UNW coordinate system vs. mean anomaly of MEO object.

Fig. a5. TLE’s prediction residual state components in UNW coordinate system vs. temporal difference of HEO object.
Fig. a6. TLE’s prediction residual state components in UNW coordinate system vs. mean anomaly of HEO object.

Fig. a7. TLE’s prediction residual state components in UNW coordinate system vs. temporal difference of GEO object.
Fig. a8. TLE’s prediction residual state components in UNW coordinate system vs. mean anomaly of GEO object.