SPACECRAFT COLLISION PROBABILITY
FOR LONG-TERM ENCOUNTERS

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ABSTRACT

This paper is concerned with the modeling of long-term spacecraft encounters for the purpose of computing collision probability. It has been observed that certain pairs of geosynchronous satellites have low relative velocity (meters per second or less), so that the time they spend in the encounter region is appreciable. During this period which can extend to well over a day, the relative velocity is no longer constant in direction nor can the combined covariance error ellipsoid be treated as constant. In contrast to the existing short-term encounter models for which the assumption of constant relative velocity is valid, a different approach to computing the collision probability must be used. In order to analyze the general case of long-term encounter, we must account for the covariance error ellipsoid changing in shape, size and orientation. Thus, we can no longer consider the three Cartesian coordinates representing the relative position uncertainty as random variables. Instead, we must choose three different random variables as a common basis so that we can meaningfully compute the collision probability. This probability is determined by evaluating a three-dimensional integral over the appropriate region corresponding to the values of the random variables. For the general case of time-dependent covariance, the volume of integration is determined by taking the envelope of the "effective ellipsoid" obtained by transforming the effective hard-body sphere through instantaneous rotations and dilations governed by the way the positional covariance error ellipsoid changes in time. The locus of the center of this instantaneous ellipsoid is obtained from the relative trajectory of the two encountering objects as well as the transformation matrices that map the time-varying covariance matrix to the common denominator. This paper formulates the spacecraft collision probability for this more complex situation.

INTRODUCTION

Most previous analyses\cite{1,2,3,4,5,6} have considered the case of short-term encounters for which the trajectories of the two objects are modeled as straight lines. From a qualitative view, the
orbiting velocities are of the order of several kilometers per second and the time spent in the encounter region is only a fraction of a second or at most a few seconds, so that the effects of gravitational force are negligible. This results in essentially rectilinear motion over a large region of tens or hundreds of standard deviations in extent. Then, the volume swept out by a hard-body sphere (with radius \( \rho = R_A \) equal to the combined radius of the two spherical satellites) is considered as a long circular cylinder extending along the direction of relative velocity from minus infinity to plus infinity. Thus, instead of having to deal with a three-dimensional integral, we need only consider a two-dimensional integral.\(^\text{[1]}\) One then further reduces this to a one-dimensional integral involving the error function\(^\text{[2,5]}\) or alternatively to a line-integral\(^\text{[3]}\) on the perimeter of the collision cross-section. Finally, one may even obtain analytical approximations\(^\text{[4,6]}\) which are accurate to three significant figures over a wide range of collision parameters. Because the two orbiting objects move at high relative velocity toward each other so that they spend very little time in the encounter region, the combined covariance error ellipsoid is treated as constant. This is typically the case with low Earth-orbiting objects and many geosynchronous satellites.

However, for some infrequent pairs of geosynchronous satellites\(^\text{[7]}\), the relative motion exhibits very low relative velocities and their relative positions trace out extremely curved lines in three-dimensional space. Because of the long periods of close proximity, the combined covariance matrix becomes time-dependent. This requires us to reformulate the collision probability in a more general way to accommodate these cases of long-term encounters. An attempt\(^\text{[8]}\) had been made to formulate this problem, but it is at variance with the formulation given in this paper.

**ANALYSIS**

1. **Choice of Random Variables**

Consider Figure 1 which illustrates a reference set of coordinate axes (x, y, z) in the physical space. In this space, there is a volume V swept out by the passage of the sphere with radius \( \rho \).
Suppose we have a 1-σ covariance ellipsoid at time $t = 0$ with principal semi-axes $(a, b, c)$ in $(x', y', z')$. Let $P$ denote the transformation matrix between corresponding vectors in these two systems, i.e.,

$$r' = P \bar{r} \quad \text{where} \quad P^T = P^{-1}.$$  \hfill (1)

Suppose at time $t$, this 1-σ covariance ellipsoid has undergone a rotation $R$ with principal directions $(x'', y'', z'')$. Let this 1-σ ellipsoid also have undergone a "stretching" $D$ (general term used here for denoting expansion or compression) of the principal semi-axes $(\alpha, \beta, \gamma)$ so that it appears as shown in the coordinate system $(x''', y''', z''')$. Then, we have the following transformations

$$r'' = R r' \quad \text{where} \quad R^T = R^{-1}.$$  \hfill (2)

$$r''' = D r'' \quad \text{where} \quad D^T = D.$$  \hfill (3)

In order to have a standard basis for comparison, it is convenient (but not necessary) to choose a coordinate system in which the 1-σ ellipsoid is mapped into the sphere with radius $\sigma$. Let this be accomplished by the scale transformation $S$

$$r^* = S r''' \quad \text{where} \quad S^T = S.$$  \hfill (4)

Thus, we have an isotropic probability density function $f^*(x^*, y^*, z^*)$ in this coordinate system $(x^*, y^*, z^*)$. In this space, there is a volume $V^*$ corresponding to $V$ swept out by the sphere of radius $\rho$.

Because the covariance matrix is time-dependent, it is meaningless to perform integration over the physical space as was done simply and harmlessly in the case of short-term encounters for which the covariance there was treated as constant and $(x, y, z)$ were chosen as the random variables. Thus, we have to devise a common frame of reference for which the probability density function (pdf) is time-independent and random variables $(x^*, y^*, z^*)$ can be defined meaningfully. Once this is accomplished, we then transform the volume $V$ to the corresponding volume $V^*$ in this new coordinate system. Finally, we evaluate the probability by integrating $f^*(x^*, y^*, z^*)$ over the volume $V^*$.

By combining all the transformations, it follows that we have

$$r^* = S D R P \bar{r}.$$  \hfill (5)

For convenience, let us define the transformation $T$ by

Figure 1 - Transformation of Coordinate Systems
so that equation (5) becomes
\[ \bar{r}^* = M \bar{r} \quad \text{where} \quad M = TRP. \] (7)
Hence, we have
\[ \bar{r} = M^{-1} \bar{r}^*; \quad \bar{r}^* = \bar{r} M^T; \quad \bar{r}^T = \bar{r}^* (M^{-1})^T. \] (8)

2. Transformation of a Sphere

We shall next transform a sphere of radius \( \rho \) with center at \((x_c, y_c, z_c)\) at time \( t \) in the physical space:

\[ (x - x_c)^2 + (y - y_c)^2 + (z - z_c)^2 = \rho^2 \] (9)
\[ (\bar{r} - \bar{x}_c)^T (\bar{r} - \bar{x}_c) = (\bar{r}^T - \bar{x}_c^T) (\bar{r} - \bar{x}_c) = \rho^2. \] (10)

Substituting equations (8a) and (8c) into (10), therefore
\[ (\bar{r}^* - \bar{x}_c^*) (M^{-1})^T (M^{-1}) (\bar{r}^* - \bar{x}_c^*) = \rho^2. \] (11)

From equations (1b), (2b) and (7b), it follows that
\[ M^{-1} = (TRP)^{-1} = P^{-1}R^{-1}T^{-1} = P^TR^T \] (12)
\[ (M^{-1})^T = (P^TR^T)^{-1} = (T^{-1})^T RP. \] (13)

Consequently, from equations (6b), (12) and (13), we obtain
\[ (M^{-1})^T (M^{-1}) = (T^{-1})^T PRPP^{-1}R^{-1}T^{-1} = (T^{-1})^T T^{-1} = (T^T)^{-1} = T^{-1} = (TT)^{-1} = (T^2)^{-1}. \] (14)

Therefore, equation (11) takes the unexpectedly simple form
\[ (\bar{r}^* - \bar{x}_c^*) (T^2)^{-1} (\bar{r}^* - \bar{x}_c^*) = \rho^2. \] (15)

Equation (15) describes an ellipsoid with principal axes along \((x^*, y^*, z^*)\) with center at the point \((x_c^*, y_c^*, z_c^*)\). This "derived ellipsoid" has been transformed from the original sphere with radius \( \rho \), and its shape depends on the time-dependent diagonal matrix \((T^2)^{-1}\). We note that the transformation matrix \( T \) involves only expansion and contraction but not rotation of the covariance ellipsoid. The effects of the covariance ellipsoid rotation matrix \( R \) show up in the transformation of the center \((x_c, y_c, z_c)\) to \((x_c^*, y_c^*, z_c^*)\) at time \( t \) through the equation
\[ \vec{r}_c^* = M \vec{r}_c \text{ where } M = T R P. \]  

(16)

Thus, we may state the following important theorem which reiterates what has already been proved:

**Theorem:** Suppose the covariance matrix \( C \) is time-dependent (involving changes in shape, size and orientation) is transformed to the \((x^*, y^*, z^*)\)-space of random variables by equation (5). Then, the sphere of radius \( \rho \) in the physical space \((x, y, z)\)-space is transformed to an ellipsoid in the \((x^*, y^*, z^*)\)-space. The principal axes of this ellipsoid are always along the \((x^*, y^*, z^*)\) coordinate axes. Its shape and size depend only on the time-dependent diagonal matrix \((T^2)^{-1}\) where \( T \), defined by equation (6), involves only expansion and contraction but not rotation.

In particular, even if the hard-body sphere does not move in the \((x, y, z)\) coordinate system but the covariance ellipsoid rotates, the center \((x_c, y_c, z_c)\) will move along a circular arc in the \((x^*, y^*, z^*)\) coordinate system. In general, the trajectory of the center depends on both the motion of the sphere and the rotation of the covariance ellipsoid as dictated by equation (16). Thus, the tangent to the trajectory is in the direction of the following vector derivative of the center position in the \((x^*, y^*, z^*)\) coordinate system

\[ \frac{d\vec{r}_c^*}{dt} = M \frac{d\vec{r}_c}{dt} + \frac{dM}{dt} \vec{r}_c. \]

(17)

It is important to note that this tangent vector is not the same as the "physical" velocity \( (d\vec{r}_c / dt) \) of the center transformed to the \((x^*, y^*, z^*)\) coordinate system. This "physical" velocity, like the position vector \( \vec{r}_c \) of the center, transforms from the \((x, y, z)\) coordinate system to the \((x^*, y^*, z^*)\) system according to equation (16) which comprises only the first term of equation (17). It is not tangent to the trajectory of the center in the \((x^*, y^*, z^*)\) coordinate system. Equation (17) contains another term resulting from the time-varying nature of the covariance. This is a very important point.

### 3. Motion of a Sphere in Physical Space

We next obtain the equations describing the motion of the center of the hard-body sphere according to the laws of dynamics. This is given by the relative motion of one satellite with respect to the other. Since we are concerned with long-term encounter between two geosynchronous satellites, we may safely use the Clohessy-Wiltshire equations as a first approximation. The reason is that their orbits are essentially circular and they are far away as to justify the validity of the Keplerian motion assumption over one day. The equations of motion in a rotating rectangular coordinate system are given in terms of the initial relative position \((x_o, y_o, z_o)\) and relative velocity \((\dot{x}_o, \dot{y}_o, \dot{z}_o)\) by (9)

\[ x_c = x_o + (6\omega y_o - 3\dot{x}_o) t + 2 \frac{\dot{y}_o}{\omega} \left( 1 - \cos \omega t \right) + \left( 4 \frac{\dot{x}_o}{\omega} - 6y_o \right) \sin \omega t \]

(18)
\[
y_c = 4y_o - 2\frac{\dot{x}_o}{\omega} + \left(2\frac{\ddot{x}_o}{\omega} - 3y_o\right)\cos \omega t + \frac{\dot{y}_o}{\omega} \sin \omega t
\]

(19)

\[
z_c = \frac{\dot{z}_o}{\omega} \sin \omega t + z_o \cos \omega t.
\]

(20)

The expressions on the RHS are used for the motion of the center \((x_c, y_c, z_c)\) of the sphere in the \((x, y, z)\)-space. By using the transformation (16), we obtain the trajectory of \((x^*, y^*, z^*)\) in the \((x^*, y^*, z^*)\)-space. Even though we use equations (18) through (20) subsequently, they are not easily amenable to simple physical interpretation.

We shall use rewrite them in the "phase-amplitude" form frequently used for expressing signals in electrical engineering:

\[
x_c = C + Dt + A \sin(\omega t + \phi)
\]

(21)

\[
y_c = \frac{2D}{3\omega} + \frac{1}{2} A \cos(\omega t + \phi)
\]

(22)

\[
z_c = B \cos(\omega t - \psi).
\]

(23)

The six parameters corresponding to the initial relative position and relative velocity are now replaced by the constant \(C\), the drift \(D\), the amplitude \(A\) for the \(x\) and \(y\) motion, the amplitude \(B\) for the \(z\) motion, the phase \(\phi\) for the \(x\) and \(y\) motion, and the phase \(-\psi\) for the \(z\) motion. Note that the amplitude of the \(y\) motion in equation (22) is one-half of that of the \(x\) motion in equation (21). It may be shown that these parameters are related by

\[
A^2 = \left(2\frac{\dot{y}_o}{\omega}\right)^2 + \left(4\frac{\dot{x}_o}{\omega} - 6y_o\right)^2
\]

(24)

\[
B^2 = (z_o \omega)^2 + (\dot{z}_o)^2
\]

(25)

\[
C = x_o + 2\frac{\dot{x}_o}{\omega}
\]

(26)

\[
D = 6y_o \omega - 3\dot{x}_o
\]

(27)

\[
\phi = \tan^{-1}\left(\frac{-2\frac{\dot{y}_o}{\omega}}{4\frac{\dot{x}_o}{\omega} - 6y_o}\right)
\]

(28)

\[
\psi = \tan^{-1}\left(\frac{-z_o \omega}{\dot{z}_o}\right)
\]

(29)
Figure 2 illustrates an instantaneous (x,y)-profile of the relative orbit without any drift motion. Note that it is an ellipse with 2:1 aspect ratio. Under no circumstances can it be a circle.

If there is drift (D > 0) along the positive x-direction, then the center of the osculating ellipse moves in the radial-intrack plane as shown in Figure 3.
Similarly, the center of the osculating ellipse moves in the intrack-crosstrack plane as shown in Figure 4.

Figure 3. Radial and Intrack Profile of Relative Orbit.

Figure 4. Intrack and Crosstrack Profile of Relative Orbit.

4. Volume of Integration

Finally, it remains to discuss how we obtain the volume of integration $V^*$ for computing the probability of collision between the two satellites. If we substitute equations (18) through (20) into (16), then we obtain $(x_c^*, y_c^*, z_c^*)$ as functions of time $t$. Next, we substitute these into equation (15) and obtain an equation written symbolically as

$$F(x^*, y^*, z^*, t) = 0.$$  \hspace{1cm} (30)

We had noted that this "derived ellipsoid" has been transformed from the original hard-body sphere, and its shape depends on the time-dependent diagonal matrix $(T^2)^{-1}$. The trajectory of the center $(x_c^*, y_c^*, z_c^*)$ moves according to equations (16) and (18) through (20). We may consider time $t$ as a parameter characterizing a family of such ellipsoids. Over an interval of time, let a surface $E^*$ denote an envelope to this family of ellipsoids such that each of them is tangent to $E^*$ along a locus of points. We now have a problem which is the generalization of the classic case of an envelope tangent to a family of curves$^{[10]}$. For the general case, we shall use a different approach which is as follows: If we are given a family of surfaces
\[ F(x_1, x_2, \ldots, x_n, t) = 0 \] 

with \( t \) as a parameter characterizing each surface, then we may consider equation (31) as describing a "hypersurface" in the \((n+1)\)-dimensional space of the variables \((x_1, x_2, \ldots, x_n, t)\) so that if we take cuts of constant \( t \), we recover each member of the family. Next, we consider an extended family of surfaces \( F(x_1, x_2, \ldots, x_n, t) = c \) of which \( c = 0 \) is a member. Thus, for a surface \( E^* \) to be tangent to the family (31), we must have

\[ \frac{\partial}{\partial t} F(x_1, x_2, \ldots, x_n, t) = 0. \] 

(32)

It follows that the envelope \( E^* \) of the family of ellipsoids (30) has to satisfy

\[ \frac{\partial}{\partial t} F(x^*, y^*, z^*, t) = 0. \] 

(33)

By eliminating \( t \) between equations (33) and (30), we obtain the equation of the envelope \( E^* \)

\[ \Phi(x^*, y^*, z^*) = 0. \] 

(34)

Equation (34) describes a surface which bounds a volume \( V^* \) in the \((x^*, y^*, z^*)\)-space. The general nature of this surface is very complicated. It depends on both the motion of the sphere and the time variation of the covariance ellipsoid. Even if the covariance ellipsoid does not change much in time in both its dilation and rotation, the envelope \( E^* \) still depends on the motion of the center and the radius of the sphere. If the radius of curvature of the motion of the center is smaller than the radius of the sphere, this surface can cut back on itself and we have to be extremely cautious in determining the volume bounded by it. On the other hand, if the radius of curvature is large compared to the radius of the sphere, this surface resembles a gently bending tube with an undulating cross-section as depicted in Figure 5.
5. Examples of Volume $V$ and $V^*$

5.1 Torus in Volume $V$

We may represent the torus as symmetric about the $z$-axis with $r$ the radius of the circular cross-section and $R$ the radius of the circular locus of the center of the cross-section. The equation for this surface may be obtained simply, as in books on mensuration, by rotating a circle of radius $r$ with center at a distance $R$ from the $z$-axis. However, to illustrate the foregoing method, we wish to obtain this equation as the envelope of a family of spheres of radius $r$ whose center moves in a circle of radius $R$.

Thus, the family of moving spheres parameterized by time $t$ takes the form

$$(x - R \sin t)^2 + (y - R \cos t)^2 + z^2 = r^2$$

which is equivalently

$$(x^2 + y^2) - 2R(x \sin t + y \cos t) + R^2 + z^2 = r^2.$$  

By differentiating with respect to $t$ and setting to zero, we obtain the following envelope condition

$$x \cos t - y \sin t = 0$$

which yields

$$\sin t = x/\sqrt{x^2 + y^2} \quad \text{and} \quad \cos t = y/\sqrt{x^2 + y^2}.$$  

Figure 5. Envelope of a Family of Ellipsoids.
By substituting equation (38) into (36) and then regrouping terms, we obtain the envelope

$$\left(\sqrt{x^2 + y^2} - R\right)^2 + z^2 = r^2.$$ \hspace{1cm} (39)

### 5.2 Elliptical Doughnut in Volume $V^*$

We shall next consider the case of elliptical relative motion about the origin. With $C = D = 0$ in the Clohessy-Wiltshire equations (21) through (23), the motion of the center becomes

$$x_c = A \sin \tau, \quad y_c = \frac{1}{2} A \cos \tau \quad \text{and} \quad z_c = 0 \quad \text{where} \quad \tau \equiv \omega t + \phi.$$ \hspace{1cm} (40)

In this special case, the trajectory of the center is a closed curve, and this surface resembles an elliptical doughnut with a somewhat (but not exactly) elliptical cross-section.

For simplicity, we consider the case of covariance which is time-independent and isotropic so that we now have a sphere of constant radius with the center moving along the ellipse described by equation (40). Hence, the volume $V$ in the $(x, y, z)$-space is the same as the volume $V^*$ in the $(x^*, y^*, z^*)$-space. Moreover, because of this simplification, we note that in equation (7) the matrix $M$ is equal to the identity matrix $I$ so that we may omit the asterisk in this special case.

Equation (30) now takes the form

$$\left(x - A \sin \tau\right)^2 + \left(y - \frac{1}{2} A \cos \tau\right)^2 + z^2 = \rho^2$$ \hspace{1cm} (41)

which is equivalently

$$(x^2 + y^2) - 2A(x \sin \tau + \frac{1}{2} y \cos \tau) + \frac{1}{4} A^2 + \frac{3}{4} A^2 \cos^2 \tau + z^2 = \rho^2.$$ \hspace{1cm} (42)

By differentiating with respect to $\tau$, we obtain from equation (33)

$$2Ax \cos \tau - A \sin \tau \left(y + \frac{3}{2} A \cos \tau\right) = 0.$$ \hspace{1cm} (43)

By multiplying equation (42) by $\cos \tau$ and (43) by $\sin \tau$, using the identity $\sin^2 \tau + \cos^2 \tau = 1$, and then eliminating terms, we obtain the following cubic equation in $\cos \tau$

$$\frac{3}{4} A^2 \cos^3 \tau + \left(x^2 + y^2 + z^2 - \frac{1}{2} A^2 - \rho^2\right) \cos \tau - Ay = 0.$$ \hspace{1cm} (44)
This equation is already in the "reduced" form because the second degree term is absent. It may be readily solved for $\cos \tau$ by Cardan's Method\(^{[11]}\). Thus, we define

$$p = 4 \left( x^2 + y^2 + z^2 - \frac{1}{2} A^2 - \rho^2 \right)^{1/2} A^2, \quad q = -4y/3A = 0 \quad \text{and} \quad Q = \left( \frac{p}{3} \right)^{1/3} + \left( \frac{q}{2} \right)^{2/3}. \quad (45)$$

The solution is then given by

$$\cos \tau = \sqrt{\frac{q}{2} + \sqrt{Q}} + \sqrt{\frac{q}{2} - \sqrt{Q}}. \quad (46)$$

The resultant expression may then be substituted into either equation (41) or (43) to obtain the equation (34) for the envelope. Caution must be exercised to determine if the surface cuts back on itself. If the radius $\rho$ of the circular cross-section is smaller than $A/2$, then the elliptical doughnut will not cut back on itself and there will only be one hole. This surface of genus 1 is called a "torus" as shown in Figure 6. A little consideration will reveal that if $\rho \geq A/2$, then there will be no hole. This surface of genus 0 may be called a "sphere" as illustrated in Figure 7. Thus, for a sphere of constant radius with center moving along an ellipse, there cannot be a surface generated of genus 2 called a "pretzel". The foregoing classifications take after those of differential geometry except for issues pertaining to differentiability. Whatever the nature of this surface, it determines the volume of integration $V$ for the isotropic three-dimensional Gaussian pdf in order to compute the probability of collision for the case of long-term encounters. Because of the possibility of the surface intersecting itself, we cannot simply determine the probability of collision by considering cylindrical disks (with normal in the direction of the instantaneous tangent) that are covered by the motion of the center over infinitesimal periods of time. Moreover, because of the curvature of the trajectory of the center, these cylindrical slices have the additional problem of overlapping on the concave side and introducing gaps on the convex side of the curved trajectory. Thus, the principles of calculus cannot be applied even in the limit of infinitesimal slices.
Figure 6. "Torus" (Surface of Genus 1) for $\rho < A/2$.

Figure 7. "Sphere" (Surface of Genus 0) for $\rho \geq A/2$.

6. Probability of Collision
From the foregoing discussion, we compute the probability of collision $P$ by integrating the three-dimensional isotropic Gaussian pdf over the volume $V^*$ in the $(x^*, y^*, z^*)$-space

$$P = \frac{1}{(2\pi)^{3/2} \sigma^3} \iiint_{V^*} e^{-\frac{(x^* - C)^2 + (y^*)^2 + (z^*)^2}{2\sigma^2}} dx^* dy^* dz^*.$$  \hspace{1cm} (47)$$

The volume $V^*$ is bounded by the enveloping surface $E^*$ given by $\Phi(x^*, y^*, z^*) = 0$ as in equation (34). In general, this surface $E^*$ and hence the volume $V^*$ are very complicated. Moreover, we do not usually have the explicit matrices needed in equation (7) to transform from $V$ to $V^*$. Hence, we resorted to simplifying assumptions as in the example with the elliptical doughnut discussed previously. Unless we specify how the covariance changes with time, all we have at our disposal are the comparatively simple case of closed relative trajectories as shown in Figures 2 and 6. Even this meager information is sufficient to illustrate vital differences between short-term and long-term encounters. Thus, we work with equations (21) through (23) which yield

$$x_c = C + A \sin \tau, \quad y_c = \frac{1}{2} A \cos \tau \quad \text{and} \quad z_c = 0 \quad \text{where} \quad \tau = \omega t + \phi.$$ \hspace{1cm} (48)$$

Hence, we have an elliptical doughnut described by equations (41) through (44), but displaced by a constant $C$ on the $x$-axis. The integral $P$ may be evaluated numerically and this can be very time-consuming. Moreover, this approach yields little insight and has to be repeated whenever the input collision parameters such as the standard deviation $\sigma$, the spherical radius $\rho$, the trajectory semi-major axis $A$ and the displacement $C$ are changed. An alternative approach is to exploit the Method of Equivalent Cross-Section Areas (MECSA) discussed in Section 5.6 of Reference 12 which has been found to be numerically accurate to approximately three significant figures in computing the collision probability for most spacecraft encounters. We need now to make a slight change of viewpoint since we no longer apply it to collision cross-section areas but to general integration areas. For convenience, we shall refer to this as the Method of Equivalent Areas (MEA). We proceed as follows:

First, we present Figure 6 showing only the outlines of the envelope as depicted in Figure 8.
Figure 8. Outline of Envelope in (x, y)-Plane.

Even though these two outlines are not true ellipses, we shall approximate them by ellipses: The outer with semi-major axis \( (A+\rho) \) and semi-minor axis \((\frac{1}{2}A+\rho)\); the inner with semi-major axis \((A-\rho)\) and semi-minor axis \((\frac{1}{2}A-\rho)\). Thus, their respective areas are given by

\[
A_{\text{outer}} = \pi(A+\rho)(A/2+\rho) \quad \text{and} \quad A_{\text{inner}} = \pi(A-\rho)(A/2-\rho) .
\]  

(49)

We approximate each ellipse by a circle of equal area. Thus, we have

\[
R_{\text{outer}} = \sqrt{(A+\rho)(A/2+\rho)} \quad \text{and} \quad R_{\text{inner}} = \sqrt{(A-\rho)(A/2-\rho)} .
\]  

(50)

We now essentially have the situation of a circular torus with radius \(R\) from the symmetry axis (parallel to the \(z\)-axis and displaced along the \(x\)-axis by a distance \(C\)) and radius \(r\) of cross-section given by

\[
R = (R_{\text{outer}} + R_{\text{inner}})/2 \quad \text{and} \quad r = (R_{\text{outer}} - R_{\text{inner}})/2 .
\]  

(51)

This circular torus is shown in Figure 9.
Let us next consider a two-dimensional integral \( I \) over the area of the plane \( z = \beta \) within the circular torus. This \((x, y)\)-integration area is an annulus \( A' \) bounded by two concentric circles of radii \((R+\alpha)\) and \((R-\alpha)\) centered at the point \((C,0)\). It is easily shown that we have

\[
\alpha = \sqrt{r^2 - z^2}.
\]  

The integral \( I \) is given by

\[
I = \frac{1}{2\pi \sigma^2} \iint_{\text{Annulus } A'} e^{-(x^2+y^2)/2\sigma^2} \, dx \, dy. \tag{53}
\]

Let the integrals \( I_{R+\alpha} \) and \( I_{R-\alpha} \) be defined by

\[
I_{R+\alpha} = \frac{1}{2\pi \sigma^2} \iint_{\text{Circle Radius (R+\alpha)}} e^{-(x^2+y^2)/2\sigma^2} \, dx \, dy \quad \text{and} \quad I_{R-\alpha} = \frac{1}{2\pi \sigma^2} \iint_{\text{Circle Radius (R-\alpha)}} e^{-(x^2+y^2)/2\sigma^2} \, dx \, dy. \tag{54}
\]

The two-dimensional integral \( I_{R+\alpha} \) can be transformed into a one-dimensional integral involving the Rician pdf and we obtain the following result accurate to approximately three significant figures for the case of zeroth order \((m = 0)\) of approximation of the Rician integral. It can be shown that the range of applicability is \((R+r)/C/\sigma < 0.28\) for this approximation.

\[
I_{R+\alpha} = e^{-v/2} (1 - e^{-u_{R+\alpha}/2}) \quad \text{where} \quad u_{R+\alpha} = \left(\frac{R + \alpha}{\sigma}\right)^2 \quad \text{and} \quad v = \left(\frac{C}{\sigma}\right)^2. \tag{55}
\]

Similarly, the two-dimensional integral \( I_{R-\alpha} \) can be transformed into the following one-dimensional integral

\[
I_{R-\alpha} = e^{-v/2} (1 - e^{-u_{R-\alpha}/2}) \quad \text{where} \quad u_{R-\alpha} = \left(\frac{R - \alpha}{\sigma}\right)^2 \quad \text{and} \quad v = \left(\frac{C}{\sigma}\right)^2. \tag{56}
\]

Therefore, the integral \( I \) over the annulus \( A' \) is given by

\[
I = I_{R+\alpha} - I_{R-\alpha} \\
= e^{-v/2} (e^{-u_{R+\alpha}/2} - e^{-u_{R-\alpha}/2}) \\
= e^{-v/2} (e^{-(R^2-2R\alpha+\alpha^2)/2\sigma^2} - e^{-(R^2+2R\alpha+\alpha^2)/2\sigma^2}) \\
= e^{-v/2} e^{-(R^2+\alpha^2)/2\sigma^2} \left( e^{R\alpha/\sigma^2} - e^{-R\alpha/\sigma^2} \right) \\
= e^{-v/2} e^{-(R^2+r^2-2z^2)/2\sigma^2} \left( e^{R\sqrt{r^2-z^2}/\sigma^2} - e^{-R\sqrt{r^2-z^2}/\sigma^2} \right). \tag{57}
\]
Hence, the collision probability $P$ in equation (47) takes the form

$$P = \frac{1}{\sqrt{2\pi\sigma}} \int_{-\infty}^{\infty} e^{-z^2/2\sigma^2} \, dz$$

$$= \frac{2}{\sqrt{2\pi\sigma}} e^{-\nu/2} e^{-(R^2 + r^2)/2\sigma^2} \int_{0}^{\nu} \left( e^{R\sqrt{r^2 - z^2}/\sigma^2} - e^{-R\sqrt{r^2 - z^2}/\sigma^2} \right) dz$$

$$= \frac{4}{\sqrt{2\pi\sigma}} e^{-\nu/2} \int_{0}^{\nu} \sinh \left( \frac{R\sqrt{r^2 - z^2}}{\sigma^2} \right) dz.$$  

(58)

It appears at first sight that the RHS of equation (58) has to be evaluated numerically. However, a little consideration reveals that we may express the integrand as

$$\sinh \xi = \sum_{n=0}^{\infty} \frac{\xi^{2n+1}}{(2n+1)!}.$$  

(59)

Thus, we obtain the following expression

$$P = \frac{4}{\sqrt{2\pi\sigma}} e^{-\nu/2} \int_{0}^{\nu} \sum_{n=0}^{\infty} \frac{\xi^{2n+1}}{(2n+1)!} \, dz \quad \text{where} \quad \xi \equiv \frac{R\sqrt{r^2 - z^2}}{\sigma^2}. $$  

(60)

Let us introduce a new variable $\theta$ defined by

$$z = r \sin \theta \quad \text{so that} \quad dz = r \cos \theta \, d\theta. $$  

(61)

Then, it is easily shown that

$$\xi = \frac{Rr}{\sigma^2} \cos \theta, \quad \xi \, dz = \frac{Rr^2}{\sigma^2} \cos^2 \theta \, d\theta \quad \text{and} \quad \xi^{2n+1} \, dz = \frac{Rr^2}{\sigma^2} \left( \frac{Rr}{\sigma^2} \right)^{2n} \cos^{2n+2} \theta \, d\theta. $$  

(62)

By substituting equation (62) into (60) and interchanging the order of integration and summation, we obtain

$$P = \frac{4}{\sqrt{2\pi}} e^{-\nu/2} \frac{Rr^2}{\sigma^2} \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} \left( \frac{Rr}{\sigma^2} \right)^{2n} \int_{0}^{\pi/2} \cos^{2n+2} \theta \, d\theta. $$  

(63)

The integral on the RHS can be obtained in closed form$^{[11]}$

$$\int_{0}^{\pi/2} \cos^{2n+2} \theta \, d\theta = \frac{\pi}{2} \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdots (2n+1)}{2 \cdot 4 \cdot 6 \cdot 8 \cdots (2n+2)} = \frac{\pi}{2} \frac{(2n+1)!}{2^{2n+1} n!(n+1)!}. $$  

(64)
Consequently, equation (63) becomes

\[
P = \sqrt{2\pi} e^{-\frac{(C^2 + R^2 + r^2)}{2\sigma^2}} \frac{Rr^2}{\sigma^3} \sum_{n=0}^{\infty} \left( \frac{Rr}{\sigma^2} \right)^{2n} \frac{1}{2^{2n+1} n!(n+1)!}.
\]  

(65)

As stated previously, the range of applicability is \((R+r)C/\sigma^2 \leq 0.28\) for this approximation. Caution must be exercised in using equation (65) in that the range of applicability is not violated.

It is easily seen that the infinite series on the RHS is rapidly convergent for all values of the parameters \(C, R, r\) and \(\sigma\) because it is term-wise dominated by the exponential series. The combined hard-body radius \(r\) is approximately 10 m. For most long-term geosynchronous spacecraft encounters, we are probably dealing with \(R\) of the order 10 km or less. We may assume that the effective combined standard deviation \(\sigma\) is of the order of 1 km. Hence, the ratio \((Rr/\sigma^2)^2\) of successive terms in the infinite series is approximately 0.01. It probably suffices to take only the first two terms in equation (65) to obtain the probability of collision for most long-term spacecraft encounters. However, if we are dealing with \(R\) of the order of 100 km, then we may need about 4 or 5 terms. Instead of evaluating the terms individually and then summing them, it is more efficient and more accurate to use the following recursion relation

\[
T_0 = \frac{1}{2}, \quad T_n = T_{n-1} \left( \frac{Rr}{\sigma^2} \right)^2 \frac{1}{4n(n+1)} \quad \text{for} \quad n = 1, 2, 3\ldots.
\]  

(66)

DISCUSSION

Again, we reiterate that the integral solution equation (58) and the equivalent infinite series solution equation (65) correspond to the case of the zeroth order of approximation of the Rician integral. This solution is valid if \((R+r)C/\sigma^2 \leq 0.28\). In particular, it is applicable to the case of moderately large \((R+r)/\sigma\) and small \(C/\sigma\) or to the case of small \((R+r)/\sigma\) and moderately large \(C/\sigma\) as long as the inequality is satisfied. An example of the former is the case of \(\sigma = 15\) km and debris orbiting the spacecraft of radius \(r = 5\) m in a circle with radius \(R = 10\) km displaced at a distance \(C \leq 6.3\) km. An example of the latter is the case of \(\sigma = 5\) km and debris orbiting in a circle with radius \(R = 70\) m at a distance \(C \leq 50\) km from the International Space Station, whose equivalent radius \(r = 70\) m. Both of these examples are of physical interest and of practical concern. Equation (65) is not restricted to just a thin torus \((r \ll R)\), but is also applicable to thick torus \((r \approx < R)\). The first example falls under the case of a thin torus, while the second example falls under the case of a thick torus.

In the same way as we proceeded above, we may obtain higher orders of approximation of the collision probability in the volume of a general (thick or thin) torus. Orders of approximation up to the 5th order have been obtained by the author but they are very complicated in form. Instead of just one infinite series as in equation (65), we now have \((m+1)^2\) infinite series for order \(m\). Thus, for the 5th order, it can be shown that we have 36
infinite series. It can be shown that this 5th order solution is valid if \((R+r)C/\sigma^2 \leq 5.3\). In particular, for the first example above of \(\sigma = 15\) km and debris orbiting a spacecraft of radius \(r = 5\) m in a circle with radius \(R = 10\) km, we must have the displaced distance \(C \leq 119\) km in order for the solution to be valid. For the second example of \(\sigma = 5\) km and debris orbiting in a circle with radius \(R = 70\) m, we note that the solution is valid up to a distance of \(C \leq 1325\) km from the International Space Station (ISS).

In the above analysis, we used the Method of Equivalent Areas (MEA) to obtain an analytical expression for the collision probability for the comparatively simple case of an elliptical torus. We can apply this procedure to analyze more complex volumes of integration by effectively using equivalent areas as previously done in the case of complex spacecraft geometries (See Chapters 5 and 6 of Reference 12). However, we may have to resort to numerical integration, or approximate evaluation of areas, before we can convert these to circles of the same area (See Appendix B of Reference 12 as an example). Once we do this, we have essentially converted the problem to one described by the Rician pdf. Finally, we proceed to integrate along the remaining direction. There are no rigid rules which govern which two integration variables to use first and which to use last. In the above example, we used the \((x, y)\)-plane first because of the simplicity of performing that conversion to existing available results. Any other choice would have made the analysis more complicated. However, for other more complex volumes of integration, it is possible that a different choice may prove more advantageous. For enveloping surfaces which cut back on themselves, it is also possible that we may simplify the integration by breaking the volume into a number of components which are easier to handle and then apply MEA.

CONCLUSION

If the covariance error ellipsoid changes in shape, size and orientation, we can no longer consider the physical coordinates \((x, y, z)\) representing the relative position uncertainty in the physical space as random variables. Instead, we must choose three different random variables \((x^*, y^*, z^*)\) as a common basis so that we can meaningfully compute the collision probability. For instance, we can define an appropriate set \((x^*, y^*, z^*)\) referred to a specific instant of time (say \(t = 0\)) as the random variables. This will be consistent with the formulation of the probability integral involving a time-independent probability density function. Formulations based on the incremental addition of probability as a function of time and time-dependent pdf are at variance with the crux of probability theory. It is quite easy to construct examples which yield absurd results.

We advocate the use of a combination of numerical evaluation of areas and the method of equivalent areas before we can apply the Rician distribution to compute the collision probability for the case of complicated long-term encounters.

An analytical expression has been obtained for computing the probability of collision between two spacecraft whose relative motion is an ellipse (or a circle). The main results are given by equation (65) for which the range of applicability is \((R+r)C/\sigma^2 \leq 0.28\). In order to focus our attention to illustrate some vital differences between short-term and long-term encounters, it is sufficient to use the zeroth order given by equation (65). For other more
subtle differences, we may use the 5\textsuperscript{th} order solution whose range of applicability is \[(R+r)C/\sigma^2 \leq 5.3.\]

The advantages of using an analytical solution are that: (1) It eliminates the numerical evaluation of the collision integral by a computer program which may be difficult to develop logically; (2) It circumvents time-consuming Monte Carlo simulations of potential collisions which can take as much as several days to complete; and (3) It rapidly yields results if the collision parameters such as the standard deviation $\sigma$, the combined radius $\rho$, the trajectory semi-major axis $A$ and the displacement $C$ are changed without having to repeat the computer runs all over again.

**REFERENCES**


