Enveloping surface of trajectory family of isotropically ejected particles due to a satellite explosion

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Introduction

An artificial satellite explosion leads to ejection into space of a mass of particles. Typical velocities of ejecta are of the order of km/s or less, so they remain on geocentric orbits, $T$, close to the supposedly circular orbit of the satellite. The emerged swarm fills out a domain, $D$, swept by family $\{T\}$. We are interested in the structure of the domain, $D$, corresponding to the largest possible family $\{T\}$. It leads to an assumption of all-directed ejecta. From imaging we observe an isotropic ejection with all possible velocities lower in absolute value than $b$. Due to the inequality of orbital periods, fragments will densely fill the domain, $D$, in a short period of time (few days; a more precise value depends on $b$ and on the altitude of the satellite orbit). To get the boundary $S$ of $D$, it is sufficient to suppose the velocities equal to $b$. So $D$ represents a debris complex emerging in a few days after the burst. In a few months $D$ diffuses due to nodes’ and pericentres’ motion in the gravity field of an oblate planet, resulting in an axisymmetrical figure $D'$ with the boundary $S'$. Here, we obtain parametric equations of $S$, and examine its properties. Topologically $S$ is a torus with one conic point and one rectilinear constriction. We intend to do similar work with $S'$ later.

1 Trajectories family

The above mechanism is similar to one of a dust belt formation by meteoroid impacts, that was first suggested by S. Soter [1] and was described in papers [2,3]. So we may use formulae for orbits, $T$, obtained there. Let a point, $O_1$, having a negligible mass describe a Keplerian circumference around a point, $O$, of mass, $m$. At a moment, $t_0$, an isotropic ejection from $O_1$ takes place. Particles possessing infinitely small masses move in all directions with the same velocity $b>0$ relative to $O_1$. Let us find the enveloping surface, $S$, of a 2-parametric family of ellipses $\{T\}$ describing these particles. Suppose that $O_1$ and the thrown particles describe conic sections corresponding to the gravitational parameter $ae^2$, equal to the product of the gravitational constant and $m$. Let $R$ be the radius of the circular orbit $O_1$ relative to $O$; $w = \sqrt{\frac{2}{R}}$ be the circular velocity of $O_1$, $c=b/w$. Confining ourselves to ellipses, we assume

$$w + b < \sqrt{\frac{2}{R}} \iff b < \sqrt{\frac{2}{R}} - 1 \iff c < \sqrt{2} - 1. \quad (1)$$
The condition of the absence of rectilinear and retrograde motions is less restrictive

\[ b < w \iff c < 1. \]  

(2)

Let us introduce a system of non-rotating Cartesian coordinates centered in \( O \); \( x \) – axis directed to \( O_1 \) at a moment of ejection, \( y \) – axis lying in the orbital plane in the direction of motion, \( z \)-axis coinciding with area-vector of \( O_1 \) orbit. Designate \( b, \theta, \lambda \) spherical coordinates of the velocity vector of a particle \( Q \) relative to \( O_1 \).

Assume velocity modulus \( b > 0 \) fixed, point \( (\theta, \lambda) \) belonging to a unit sphere \( \Sigma \). At the initial epoch the position and velocity of an ejected particle, \( Q \), chosen by two parameters \( \theta \) and \( \lambda \), are

\[ r_0 = (R, 0, 0), \quad v_0 = (b \sin \theta \cos \lambda, w + b \sin \theta \sin \lambda, b \cos \theta). \]

(3)

Knowing the position and velocity, we easily find the orbit, \( T \), of the point, \( Q \) [4].

Supposing \( R = 1 \) (one can easily restore the scale factor, if needed), we obtain

\[ p = A^2, \quad \Omega = 0 \]

(4)

\[ \cos i = \frac{1 + c \sin \theta \sin \lambda}{A}, \quad \sin i = . \]

(5)

\[ e \cos g = A^2 - 1, \quad e \sin g = -Ac \sin \theta \cos \lambda, \]

(6)

\( p, \Omega, i, e, g \) being the parameter, longitude of ascending node, inclination, eccentricity and argument of pericenter, respectively,

\[ A^2 = (1 + c \sin \theta \sin \lambda)^2 + c^2 \cos^2 \theta > 0. \]

(7)

For the sake of continuity we suppose \( -\frac{\pi}{2} \leq i \leq \frac{\pi}{2} \). Underline that all motions are prograde, \( i < 0 \) means that longitude refers to the descending node. Under the condition (2) it is easy to see that

\[ 1 - c \leq A \leq 1 + c. \]

(8)

An important role is played by orbits of extreme inclination, \( i_{\text{extr}} = \pm \arcsin c \), corresponding to

\[ \lambda = -\pi/2, \quad \theta = \arcsin c \quad \text{and} \quad \theta = \pi - \arcsin c, \]

(9)

in both cases \( A = \sqrt{1 - c^2} \). Ultimately, position vector of \( Q \) is determined by the formulae

\[ r = r (\cos u, \cos i \sin u, \sin i \sin u), \quad r = \frac{A^2}{1 + \alpha \cos u + \beta \sin u}, \]

(10)

with \( \alpha = A^2 - 1, \quad \beta = -Ac \sin \theta \cos \lambda \). Note that \( A, i \) are expressed by \( \theta, \lambda \) according to (7,5). For extreme inclination orbits we have, as an example

\[ r = \frac{1 - c^2}{1 - c^2 \cos u} (\cos u, \sqrt{1 - c^2} \sin u, \pm c \sin u). \]
2 Enveloping surface

Relations (10) represent equations of a curve $T$ (ellipse) parametrized by a variable $u \in [0, 2\pi]$. A family $\{T\}$ is a union of any $T$ marked by two parameters $\theta, \lambda$ running a unit sphere $\Sigma: \theta \in [0, \pi], \lambda \in [0, 2\pi]$. Parametric equations of the enveloping family $\{T\}$ of surface $S$ are given [5] by relations (10) and

$$\Phi(u, \theta, \lambda) = 0,$$

with

$$\Phi \left( \frac{\partial r}{\partial u}, \theta, \lambda \right) = 0.$$  

Equation (11) determines $u$ as an implicit function of $\theta, \lambda$. We succeeded in resolving it and finding an explicit function $u = F(\theta, \lambda)$. Substitution of $F(\theta, \lambda)$ instead of $u$ in (10) leads to the desired parametric equations of $S$. Direct calculations (better to use tools of computer algebra, "Mathematica", for example) give:

$$\Phi = \frac{2c^2r^4}{A^4} \sin \theta \sin u \sin \frac{u}{2} \Phi_1,$$

with

$$\Phi_1 = A(\sin \theta \sin \lambda + c) \cos \frac{u}{2} - 2 \sin \theta \cos \lambda \sin \frac{u}{2}.$$  

The first factor on the right-hand-side of (13) is bounded above and away from zero. The second one is due to the singularity of spherical coordinates at $\sin \theta = 0$. It disappears, after transition to non-singular, in the vicinity of the poles of the coordinates, such as $(\sin \theta \cos \lambda, \sin \theta \sin \lambda)$. Double root $u = 0$ of the equation (11) corresponds to a conic point $(1, 0, 0)$ of the surface, $S$: all orbits, $T$, pass through the point of ejection. The root $u = \pi$ corresponds to a constriction

$$\frac{1 + 2c + c^2}{1 - 2c - c^2} \leq x \leq \frac{1 - 2c + c^2}{1 + 2c - c^2}, \quad y = 0, \quad z = 0$$  

of the surface, $S$: all orbits, $T$, pass through the line of nodes. In the vicinity of the constriction, $S$ has the topology of a cylinder, $x^2 + y^2 = 1$, on the directrix, of which points $(x, y, 0)$ and $(x, -y, 0)$ are identified. Function $\Phi_1$ is a trigonometric polynomial of first degree with respect to $u$. Its coefficients vanish together at two points (9) of the sphere $\Sigma$. That means that both orbits of extreme inclination belong to $\Sigma$ entirely. In all other cases the equation (11) is equivalent to

$$\operatorname{tg} \frac{u}{2} = \frac{A \sin \theta \sin \lambda + c}{2 \sin \theta \cos \lambda},$$

with
from which we determine uniquely the cosine and sine $u$

$$\begin{align*}
\cos u &= \frac{4\sin^2 \theta \cos^2 \lambda - A^2 (\sin \theta \sin \lambda + c)^2}{4\sin^2 \theta \cos^2 \lambda + A^2 (\sin \theta \sin \lambda + c)^2}, \\
\sin u &= \frac{4 A \sin \theta \cos \lambda (\sin \theta \sin \lambda + c)}{4\sin^2 \theta \cos^2 \lambda + A^2 (\sin \theta \sin \lambda + c)^2}.
\end{align*}
$$

(16)

Draw the final parametric equations of the enveloping surface $S$

$$x = \frac{h_1}{h}, \quad y = \frac{h_2}{h}, \quad z = \frac{h_3}{h},$$

(17)

with

$$
\begin{align*}
h_1 &= 4 \sin^2 \theta \cos^2 \lambda - A^2 (\sin \theta \sin \lambda + c)^2, \\
h_2 &= 4 \sin \theta \cos \lambda (\sin \theta \sin \lambda + c)(1 + c \sin \theta \sin \lambda), \\
h_3 &= 4 c \cos \theta \sin \theta \cos \lambda (\sin \theta \sin \lambda + c), \\
h &= (2 - A^2)(\sin \theta \sin \lambda + c)^2 + 4 \sin^2 \theta \cos^2 \lambda (1 - c \sin \theta \sin \lambda - c^2).
\end{align*}
$$

Substitution $(\theta, \lambda) \mapsto (\pi - \theta, \lambda) \mapsto (\pi - \theta, \pi - \lambda)$ shows symmetry of $S$ with respect to planes $xy$ and $xz$. Conic point $(1,0,0)$ corresponds to a curve $\sin \theta \sin \lambda + c = 0$ on the sphere of parameters. Constriction (15) corresponds to a curve $\sin \theta \cos \lambda = 0$.

3 Conclusion

Surface $S$ is examined completely. It represents a topological torus with one point-like constriction and one segment-like constriction. Vicinities of the constrictions are shown on figures 3 - 6. Image of the surface $S$ is placed on fig. 7 ($c=0.05$) and 8 ($c=0.25$). The geometry of ejection and the sphere of parameters are represented on figures 1 and 2.

References