Numerical-Analytical algorithm for prediction of satellite motion in high eccentricity orbits

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The paper considers the design of the algorithm for prediction of the motion of Earth satellites, moving in orbits with high eccentricities ($e \approx 0.3 \div 0.85$).

The algorithm has no formal limitations on the values of the eccentricity. Changes in the requirements for the maximum value of the eccentricity lead to replacement of certain calculation blocks of the program, which are generated in the form of source codes using special programs generating required texts.

The algorithm is based on the perturbation theory in the form of Hori – Deprit.

For calculation of the perturbations from the Tesserals we suggest the technique using representation of the perturbations in the form of polynomials in the powers of true anomaly. The resonance branch for arbitrary resonance is implemented as well.

For the calculations of the gravitational perturbations from the Moon the averaging is performed in both motions – these of the satellite and the perturbing body. The technique for representing Lunar spherical functions in the form of spline-polynomials with their further averaging is suggested.

For the gravitational perturbations from the Sun the standard averaging scheme (only along the orbit of the satellite) is used.

The averaging takes into account the cross-influence of the second zonal harmonics and the terms, accounting of Moon gravity.

The tests demonstrated the marked superiority (regarding computation rate) of the suggested algorithm compared to numerical procedures.

§ 1 Introduction

The computer code created on the basis of the considered algorithm was successfully used for applied tasks.

The design of the algorithm has been significantly updated and corrected during trail operations. In the course of several years important experience has been gained. This experience is implemented in the suggested version of the procedure.

The solutions, resulting in the enhancements of the previous versions of the algorithm are discussed in detail. These solutions are not obvious and are based on the analysis of the performance of the algorithm.

The first two sections – §§ 2, 3 consider the fundamentals of the algorithm. Sections §§ 4 – 7 describe the calculation techniques for specific perturbing factors. Section § 8 treats the history of algorithm development. The Appendix considers the mathematical fundamentals of the procedure in more detail.

The numbering of the formulas in the paper uses two numbers: the first one – the number of the section, the second one – the number of the formula in the section, e.g. (2.1) – is the first formula in the second section.

We use the following acronyms:

1. SES – System of Element Sets of a satellite.
2. ES – Element Set (a point in SES coordinate space).
3. GUCF – Geocentric Unmovable Coordinate Frame.
4. OCF – Orbital Coordinate Frame.
5. C20 – second zonal harmonic of geopotential.
6. DADEM – Doubly Averaged Differential Equations of Motion.
7. OCM – On-going Calculation Mode.

§ 2 Perturbation theory application

Semi-analytical procedures for solving differential equations of motion use perturbation theory techniques. The following techniques have extensive applications:
Krylov-Bogoluybov-Mitropolsky [1], Delaunay - Zeipel - Poincare [2], [3], [4], [5] and Hori - Deprit [6], [7], [8]. A set of works on the demonstration of their equivalence exists. We shall not consider these issues now. Regarding the outcome for the practice, all these techniques, being, however, different lead to the same results.

We use the Hori – Deprit\(^1\) technique for the following reasons:

1. The formulas of the method have compact non-coordinate representation.
2. When the specific system of element sets (SES) is chosen, the formulas for the calculations are derived from the general formulas rather simply.

The formulas of Hori – Deprit method use Poisson brackets (see Appendix).

In the coordinate frame the Poisson brackets of two functions is represented via partial derivatives of these function:

\[
[H_1,H_2] = (\text{grad } H_1)^T \times M \times \text{grad } H_2, 
\]

where \(\text{grad } H = (\partial H/\partial x_1, \ldots, \partial H/\partial x_n)^T\) – gradient of function \(H\) (vector-column), \(M\) – skew-symmetric matrix \(n \times n\) \((n\) – even).

Further follows the representations of matrix \(M\) for several SES.

**Kepler’s SES**

\[
M, a, i, \Omega, e, \omega.
\]

**Matrix \(M\) for Kepler’s SES**

\[
\begin{array}{cccc}
0 & k_0 & 0 & 0 \\
- k_0 & 0 & 0 & 0 \\
0 & 0 & 0 & - k_2 \\
- k_4 & 0 & 0 & 0 \\
0 & 0 & k_3 & 0 \\
0 & 0 & 0 & k_1 \\
\end{array}
\]

\(\text{grad } H = (\partial H/\partial M, \partial H/\partial a, \partial H/\partial i, \partial H/\partial \Omega, \partial H/\partial e, \partial H/\partial \omega)^T,\)

\[k_0 = 2 \cdot L / \mu_E, k_1 = \eta / (e \cdot L), k_4 = \eta \cdot k_1,\]

\[k_2 = 1 / (L \cdot \eta \cdot \sin(i)), k_3 = \cos(i) \cdot k_2,\]

where \(L = \sqrt{\mu_E \cdot a}, \eta = \sqrt{1 - e^2}, \mu_E\) – Earth gravitational constant.

**Modified Delaunay SES**

\[
\lambda = M + \omega, L = \sqrt{\mu_E \cdot a}, \theta = \cos(i), \Omega, h = e \cdot \sin(\omega), k = e \cdot \cos(\omega).
\]

**Matrix \(M\) for modified Delaunay SES**

\[
\begin{array}{cccc}
0 & 1 & -c_3 & 0 \\
-l & 0 & 0 & 0 \\
c_3 & 0 & 0 & -c_1 \\
0 & 0 & c_1 & 0 \\
h \cdot c_2 & 0 & -k \cdot c_3 & 0 \\
k \cdot c_2 & 0 & h \cdot c_3 & 0 \\
\end{array}
\]

\(\text{grad } H = (\partial H/\partial \lambda, \partial H/\partial L, \partial H/\partial \theta, \partial H/\partial \Omega, \partial H/\partial h, \partial H/\partial k)^T,\)

\[c_0 = \eta / L, c_1 = 1 / (\eta \cdot L), c_2 = c_0 / \eta_1, c_3 = \theta \cdot c_0,\]

where \(\eta_1 = 1 / (1 + \eta).\)

\(^1\) The description of the method is given in Appendix (§ 9).
Modified Lagrange SES

\( \lambda_i = M + \pi, L = \sqrt{\mu^E \cdot a}, p = \sin(\pi/2) \cos(\Omega_i), q = \sin(\pi/2) \sin(\Omega_i), \)

\( h_i = e \cdot \sin(\pi), k_i = e \cdot \cos(\pi), \) where \( \pi = \omega + \Omega_i. \)

Matrix \( \mathbf{M} \) for the modified Lagrange SES

\[
\begin{array}{cccc}
0 & 1 & -p \cdot c_i' & -q \cdot c_i' & -h_i \cdot c_i' & -k_i \cdot c_i' \\
-1 & 0 & 0 & 0 & 0 & 0 \\
c_i' \cdot p & 0 & 0 & c_i' / 2 & c_i' \cdot p \cdot k_i & -c_i' \cdot p \cdot h_i \\
c_i' \cdot q & 0 & -c_i' / 2 & 0 & c_i' \cdot q \cdot k_i & -c_i' \cdot q \cdot h_i \\
c_i' \cdot h_i & 0 & -c_i' \cdot k_i \cdot p & -c_i' \cdot k_i \cdot q & 0 & -c_0 \\
c_i' \cdot k_i & 0 & c_i' \cdot h_i \cdot p & c_i' \cdot h_i \cdot q & c_0 & 0
\end{array}
\]

grad \( H = (\partial H / \partial \lambda_i, \partial H / \partial L, \partial H / \partial p, \partial H / \partial q, \partial H / \partial h_i, \partial H / \partial k_i)^T, \)

\( c_0 = \eta / L, c_1 = 1 / (2 \cdot \eta \cdot L), \) \( c_2 = c_0 / \eta_1. \)

For the Keplerian system and the modified Delaunay and Lagrange systems, the equations of Hori – Deprit method have the form (see Appendix)²:

\[
H^*_{0} = H_{0} = -\mu^E/(2 \cdot L) + T,
\]

\[
n \cdot \partial W_i / \partial M + \partial W_i / \partial t = H_i - H^*_{1},
\]

\[
n \cdot \partial W_{2} / \partial W_{1} + \partial W_{2} / \partial t = 2 \cdot (([H, W_{1}]/2 + [H^*_{1}, W_{1}]/2) - H^*_{2}),
\]

\[
n \cdot \partial W_{3} / \partial W_{2} + \partial W_{3} / \partial t = 3 \cdot (-(H^*_{1} W_{1}, W_{1})/6+2[H^*_{2} W_{1}]/3 - ([H_{2} - H^*_{2}], W_{2})/6+[H_{1} - H^*_{1}], W_{2})/2 - H^*_{3}),
\]

\[
\ldots
\]

where \( n = \mu^2 / L^3 \) – satellite mean motion, \( \mu = \mu^E \) – Earth gravitational constant.

The functions \( H^*_1, \) usually are sought for in the form, providing that the solutions of the equations (2.5) do not include secular terms. This is achieved by the averaging:

\[
H^*_1 = I (H_1),
\]

\[
H^*_2 = I ([H_1, W_1]/2 + [H^*_1, W_1]/2),
\]

\[
H^*_3 = I (-([H^*_1, W_1], W_1)/6+2[H^*_2, W_1]/3 - ([H_{2} - H^*_{2}], W_{2})/6+[H_{1} - H^*_{1}], W_{2})/2, \)

\[
\ldots
\]

where \( I \) – averaging operator.

In our procedure the averaging operator is determined by a set of rules defining the correspondence between the function \( \tilde{F} \) and the function \( \tilde{F}^* \). These rules will be specified for each perturbing factor.

\section*{Solution of prediction task}

The algorithm solves the prediction task in two stages (see § 8). The first stage determines the generator of short-periodic perturbations \( W \) (see Appendix) and singly averaged Hamilton’s function \( H^*. \) For singly averaged Hamilton’s function the generator of long-periodic perturbations \( W^* \) and doubly averaged Hamilton’s function \( H^{**} \) are determined.

The right sides of doubly averaged differential equations of motion (DADEM), obtained on the basis of \( H^{**} \) have slow temporal evolution and allow the integration with the step up to ten days long. The general solution of the prediction task can be written in the form:

\[
E = E^* + \Delta E (E^*, t),
\]

² For the modified Delaunay system we should write $\lambda$ instead of $M$, and for the modified Lagrange system - $\lambda_i$.  

3
where \( E^* = E^{**} + \Delta E^* ( E^{**} , t ) \),
\( E \) – osculating ES,
\( E^* \) – once averaged ES,
\( E^{**} \) – doubly averaged ES,
\( \Delta E \) – short-periodic perturbations, found using function \( W \),
\( \Delta E^* \) – long-periodic perturbations, found using function \( W^* \).

The operator of double averaging used for the averaging of singly averaged Hamilton’s function \( H^* \), further will be denoted as \( I^* \):
\[ H^{**} = I^* \left( H^* \right). \]

Thus the doubly averaged Hamilton’s function \( H^{**} \) is the result of successive application of the averaging operator \( I \) and the operator of secondary averaging \( I^* \) to the initial Hamilton’s function \( H \):
\[ H^{**} = I^* \left( I \left( H \right) \right). \]

§ 3 Model of Perturbing Factors

The algorithm accounts of the following perturbing factors:
1. Perturbations from geopotential:
   1.1) Zonal harmonics,
   1.2) Tesseral harmonics.
3. Sun gravity.

Keplerian motion is taken as non-perturbed model (see § 8). The force function can be represented as the sum:
\[ U = U_0 + U^Z + U^T + U^M + U^S, \]
where \( U_0 \) – force function of Keplerian motion,
\( U^Z \) – gravitational perturbations from zonal harmonics,
\( U^T \) – gravitational perturbations from the Tesserals,
\( U^M \) – Lunar gravitational perturbations,
\( U^S \) – Solar gravitational perturbations.

The relevant Hamiltonian is written in the form (see Appendix):
\[ H = H_0 + H^Z + H^T + H^M + H^S, \]
where \( H_0 = K - U_0 + T = T - \frac{\mu E}{2a} \),
\( K \) – kinetic energy, \( T \) – the variable, dual to time \( t \),
\( H^Z = -U^Z, H^T = -U^T, H^M = -U^M, H^S = -U^S \).

Algorithm construction uses the standard form for writing the geopotential. In the equatorial coordinate frame the geopotential has the form:
\[ U^E = U_0 + U^Z + U^T, \]
\( U_0 = \mu^E / r \),
\[ U^Z = -\left( \frac{\mu^E}{r} \right) \sum_{n=2}^{\infty} J_n \left( \frac{R^E}{r} \right)^n \cdot P_n \left( \sin(\delta) \right), \]
\[ U^T = \left( \frac{\mu^E}{r} \right) \sum_{n=2}^{\infty} \sum_{q=1}^{\infty} \left( \frac{R^E}{r} \right)^n \cdot P_n(\delta) \cdot \left\{ C_{nq} \cdot \cos(q(\alpha - S)) + S_{nq} \cdot \sin(q(\alpha - S)) \right\}. \]

\(^3\) This paper does not treat the atmospheric drag.
where \( r \) – distance to the point, \( \alpha \) - declination, \( \delta \) - right ascension,

\((\alpha, \delta, r)\) – spherical coordinates,

\( S \) – Greenwich sidereal time,

\( J_n, C_{nq}, S_{nq} \) – non-dimensional coefficients,

\( P_n \) – Legendre polynomials

\( P_n^{(q)} \) – Legendre associated functions.

The Moon and the Sun are considered as material points. The mass of the satellite is considered negligible (compared to the Earth, the Moon and the Sun). Under these assumptions for the inertial coordinate system we have:

\[
U^M = \mu^M \cdot \left( 1/\Delta M - (r, r_M) / r_M^3 \right), \quad (3.3)
\]

\[
U^S = \mu^S \cdot \left( 1/\Delta S - (r, r_S) / r_S^3 \right),
\]

where \( \mu^M, \mu^S \) – gravitational constants of the Moon and the Sun,

\( \Delta M = |r - r_M|, \quad \Delta S = |r - r_S|, \quad r_M = |r_M|, \quad r_S = |r_S|, \)

\( r \) – radius-vector of the point,

\( r_M \) – radius-vector of the Moon,

\( r_S \) – Radius-vector of the Sun.

§ 4 Perturbations from zonal harmonics

The terms of geopotential expansion (3.2) are convenient in complex form:

\[
U^E = U_0 + \sum_{n=2}^{\infty} \sum_{q=1}^{\infty} \Re \varepsilon (Z_{nq}^E), \quad n = 2 + \infty, \quad q = 1 + n, \quad (4.1)
\]

where \( Z_{nq}^E = J_{nq} (\mu^E / r) (R_E / a) (P_n^{(q)} (\sin(\delta)) \exp(\sqrt{-1} \cdot q \cdot (\alpha - S)), \)

\( \Re \varepsilon \) – real part of complex number,

\( \exp \) – exponential function, \( \sqrt{-1} \) – imaginary unit,

\( J_{nq}, \lambda_{nq} \) – geopotential constants.

We will use the expansion [11] :

\[
Z_{nq}^E = (-1)^{n-q} \cdot \gamma_{nq} (a/r)^{n+1} \sum_{k=-n}^{n} A_{nq}^k(i) \cdot \exp(\sqrt{-1} \cdot (k \cdot u + q \cdot \Omega - q \cdot S - q \cdot \lambda_{nq})), \quad (4.2)
\]

\( n - k \) - even ,

\( \gamma_{nq} = J_{nq} (\mu^E / a) (R_E / a)^n \)

\( A_{nq}^k(i) \) – inclination functions,

\( u = f + \omega \) - latitude argument, \( f \) – true anomaly.

For zonal harmonics \( q = 0 \) and we can write (4.2) in the form :

\[
Z_{n0}^E = (-1)^{n/2} \gamma_{n0} (a/r)^{n+1} \sum_{k=-n}^{n} A_{n0}^k(i) \cdot \exp(\sqrt{-1} \cdot k \cdot u), \quad n - k \) - even, \quad (4.3)

\( A_{n0}^k(i) = A_{n0}^k(i) \).
For the zonal harmonics the equations (2.5) have a simplified form (since the right sides are time-independent): \( n \partial W/\partial M = F \). Determination of the short periodic perturbations of the first order is reduced to solving the equations:

\[
\begin{align*}
n \partial V_{Zn}^2 / \partial M &= - (Z_{n0}^E - I(Z_{n0}^E)), \quad n = 2 + \infty \\
\text{where } W_{Zn} &= \mathbb{R}_Z (V_{Zn}^2) \text{ is the generator of the short periodicals of the first order from the zonal harmonic with index } n,
\end{align*}
\]

\[
I(F) = \frac{1}{2 \pi} \int_0^{2 \pi} F(M) dM - \text{ averaging operator,}
\]

\[
W_{Zn} = W_{Z2} + W_{Z3} + W_{Z4} + W_{Z5} + \ldots - \text{generator of the short periodicals from zonal harmonics.}
\]

Using the relationship \( dM/df = (1/\eta)(r/a)^2 \) we can easily get:

\[
\psi_n = \int Z_{n0}^E (M) dM = (1/\eta) \int Z_{n0}^E(f) (r/a)^2 df = (-1)^{n/2} \gamma_{n0} \sum_{k=-n}^{n} A_{nk}^{k}(i) \exp(\sqrt{-1} k \omega) \int (1/\eta) (a/r)^{(n-1)} \exp(\sqrt{-1} kf) df,
\]

where \( \int \) - indefinite integral, \( n - k \) - even.

Applying the expansion [11]:

\[
(p/r)^n = \sum_{m=-n}^{n} M_{nm} (e) \exp(\sqrt{-1} mf),
\]

where \( p = a \cdot (1-e^2) - \text{focal parameter of the orbit,} \)

\( M_{nm} = M_{nm} (e) - \text{eccentricity functions,} \)

\( M_{nm} = (e/2)^n \cdot \sum_j C_{n+j}^{m+2j} C_{(m+2j),j} (e/2)^{2j}, \quad j = 0 + [(n-m)/2], \)

where \( [ \cdot ] \) - entire part,

\( C_n^k = n/(n-1) \ldots(n-m+1)/m! - \text{binomial coefficients,} \)

\( m! = 1 \cdot 2 \ldots m \) - factorial.

Using (4.6) and performing integration in (4.5) yield the expression for the integral \( \psi_n \):

\[
\psi_n = \psi_n^* + \psi_n^* \cdot f, \quad (4.7)
\]

\[
\psi_n^* = (-1)^{(n-1)/2} \gamma_{n0} \sum_{k=-n}^{n} A_{nk}^{k}(i) \exp(\sqrt{-1} k \omega) \cdot \sum_{p=-(n-k)+1}^{(n-k)-1} X_n^2 p (e) \exp(\sqrt{-1} pf) / p, \quad n - k \text{ - even,}
\]

\[
\psi_n^* = \frac{2 \pi}{(2 \pi)} \int_0 Z_{n0}(M) dM = (-1)^{n/2} \gamma_{n0} \cdot \sum_{k=-(n-2)}^{n-2} A_{nk}^{k}(i) \cdot X_n^2 (e) \exp(\sqrt{-1} k \omega), \quad n - k \text{ - even,}
\]
\[
X^Z_{nk} = (1/n^2) \cdot M(n-1,p-k) (e) - eccentricity functions of zonal harmonics,
\]
\[
X^Z_{n,k} = X^Z_{nk} \cdot X^Z_{nk} = (1/n^2) \cdot M(n-1,k) (e)
\]

From (4.4), (4.5) and (4.7) we get:

\[
I(Z_{n0}) = \psi_n^*, \quad (4.8)
\]
\[
V_{1n} = - (\psi_n^- + \psi_n^* \cdot (f - M)) / n,
\]
\[
W_{1n} = \Re (V_{1n}^Z), H_{1n}^Z = - \Re (\psi_n^*).
\]

The formulas (4.7) + (4.8) give the complete solution of the first order for the zonal harmonics and are used in the algorithm. They are applicable to orbits with arbitrary eccentricities \( e < 1 \).

For the zonal harmonics we use the term of the second order of the averaged Hamilton’s function, which is obtained for the second zonal harmonic. This term has a compact form (see [12]):

\[
H^*_{n0} = - (j_2^2 \cdot \mu^E / (2a)) \cdot \{ A_1 / \eta^4 + A_2 / \eta^6 + A_3 / \eta^8 + A_4 e^2 \cdot \cos(2 \cdot \omega) / \eta^7 \}
\]

where \( j_2 = J_2 \cdot (R^E / a)^2 \),
\[
A_1 = (3/32) \cdot (5 - 18 \cdot \theta^2 + 5 \cdot \theta^4),
\]
\[
A_2 = (3/8) \cdot (1 - 6 \cdot \theta^2 + 9 \cdot \theta^4),
\]
\[
A_3 = (3/32) \cdot (-1 + 2 \cdot \theta^2 + 7 \cdot \theta^4),
\]
\[
A_4 = (3/32) \cdot (1 - 16 \cdot \theta^2 + 15 \cdot \theta^4),
\]
\[
\theta = \cos(i).
\]

§ 5 Perturbations from Tesseral harmonics

Determination of short – periodic perturbations for the orbits with high eccentricities is one of the most sophisticated tasks in the construction of the algorithm.

The classical expansion of the geopotential into trigonometric series in the multiples of mean anomaly has extremely slow convergence for high eccentricities.

Let us analyze the Tesseral harmonics as functions of mean anomaly. We will consider two orbits:

**General orbital parameters.**

\[
H_p = 300 \text{ km} - \text{perigee altitude}, \quad \omega = 30^\circ, \quad i = 63^\circ, \quad \Omega = 120^\circ, \quad \lambda = 2.454 \text{ rad}
\]

1. **Orbit No 1** : \( a = 15000 \text{ km} \Rightarrow e = 0.555 \) - moderate eccentricity

2. **Orbit No 2** : \( a = 28000 \text{ km} \Rightarrow e = 0.76 \) - high eccentricity

Fig. 1 allows to evaluate the complication of the expansion of the Tesseral harmonics into the series in the multiples of mean anomaly for high eccentricities. This is connected with the uneven behavior of the perturbing function in the vicinity of the perigee.

The dependence of perturbing function on true anomaly is smooth. However, we face the problem of solving the equations (2.5) for the generator of short periodic perturbations. The solution turns out to be much more sophisticated than for the case of expanding the perturbing function into trigonometric series in the multiples of mean anomaly.

\[\text{The plots present the perturbations from the second Tesseral harmonic.}\]
Let us consider the technique of calculating short periodicals as functions of true anomaly, used in the algorithm. The basic formulas of this technique are derived further.

We write the force function of the Tesseral harmonics in the form:

\[ U_T = \sum_{q=1}^{\infty} U_T^q \]

\[ U_T^q = \Re (Z_T^q) \]

\[ Z_T^q = T_q \cdot \exp(-\sqrt{-1} \cdot q \cdot S) \]

\[ T_q = \sum_{n=q}^{\infty} T_{nq} \]

\[ T_{nq} = J_{nq} \cdot (\mu / r) \cdot (R^E / r)^n \cdot P_n^{(q)}(\sin(\delta)) \cdot \exp(\sqrt{-1} \cdot q \cdot (\alpha - \lambda)) \]

Then determine the short periodic perturbations of the first order for the Tesseral harmonics. To do it, we have to solve the equation (2.5):

\[ n \cdot \partial W_1^T / \partial M + \partial W_1^T / \partial t = - (U_T - U_T^*) \]

The solution is reduced to solving the set of equations, written in complex form:

\[ n \cdot \partial V_T^q / \partial M + \partial V_T^q / \partial t = - (Z_T^q - Z_T^q^*) \]

\[ W_1^T = \sum_{q=2}^{\infty} \Re (V_T^q) \]

Let us consider the solution of the equations (5.2) for non-resonance and resonance cases.

**Non-resonance case**

For the non-resonance case the averaging operator I is zero: \( Z_T^q^* = I(Z_T^q) = 0 \)

In this case the solution of equation (5.2) can be written in the form:

\[ V_T^q = - \psi_q(M) \cdot \exp(-(-1)^{1/2} \cdot q \cdot S) \]

\[ \psi_q(M) = \exp((-1)^{1/2} \cdot \nu \cdot M) \cdot \left( \int_0^M T_q(\xi) \exp(-\sqrt{-1} \cdot \nu \cdot \xi) d\xi + C_q \right) \]

where \( \nu = q \cdot n^E / n \), \( n^E \) – the angular velocity of Earth rotation.

The value of the constant \( C_q \) is chosen to make the function \( \psi_q(M) \) periodical:

\[ C_q = \frac{2 \cdot \pi}{\int_0^M T_q(\xi) \cdot \exp(-\sqrt{-1} \cdot \nu \cdot \xi) d\xi} \]

(5.4)

**Resonance case**
If the denominator $\exp(-\sqrt{-1} \cdot 2\pi \cdot \nu) - 1$ in the expression (5.4) is small, we are dealing with resonance case. Let us define the averaging operator for this case:

$$I(\bar{Z}_q) = Z_q^* =$$

$$\exp(-1/p\cdot M) \cdot \int_0^{2\pi} Z_q(M, S) \cdot \exp(-\sqrt{-1} \cdot p\cdot M) \, dM / (2\pi) = C_q \cdot \exp(-1/p\cdot M - q\cdot S)$$

where $C_q = \int_0^{2\pi} \exp(-1/p\cdot M) \, dM / (2\pi)$

$p$ – entire number, closest to $\nu$

$Z_q^*$ – correspond to the resonance term in the expansion of $Z_q^*$ into trigonometric series in the multiples of mean anomaly.

Solution of the equation (5.2) in this case is written in the form:

$$V_q^* = -/(\psi_q(M) - C_q \cdot \exp(-\sqrt{-1} \cdot p\cdot M) / \delta \cdot \exp(-\sqrt{-1} \cdot q\cdot S)) \cdot \exp(-1/p\cdot M) \cdot dM / (2\pi)$$

where $\delta = \sqrt{-1} \cdot (p - \nu)$

Eliminating the singularities $1/\delta$, $1/(\exp(-\sqrt{-1} \cdot 2\pi \cdot \nu) - 1)$ we obtain the formula:

$$V_q^* = -X_q(M) \cdot \exp(-\sqrt{-1} \cdot q\cdot S) \cdot \exp(-1/p\cdot M) \cdot dM / (2\pi)$$

where $X_q(M) = \exp(-\sqrt{-1} \cdot \nu\cdot M) \cdot \int_0^M T_q(M) \cdot \exp(-\sqrt{-1} \cdot \nu\cdot M) \cdot dM$

$$\int_0^{2\pi} \cdot \int_0^M T_q(M) \cdot \exp(-\sqrt{-1} \cdot \nu\cdot M) \cdot dM / (2\pi)$$

where $\delta = 2\pi \cdot \delta$

$\phi_1(x) = 1/(\exp(x) - 1) - 1/x = -1/2 + x/12 + x^3/720 + ...,$

$\psi_1(x) = (\exp(x) - 1) / x = 1 + x/2 + x^2/6 + x^3/24 + x^4/120 + ...,

The function $X_q(M)$ is periodical, its formula does not include small denominators.

Solution of the equation (5.2) uses the integrals of the form:

$$I\psi(g) = \int_0^M g(\xi) \cdot \exp(-\sqrt{-1} \cdot \nu\cdot \xi) \, d\xi$$

We used spline - polynomials of true anomaly for representation of functions $I\psi(g)$.

The plots presented in Fig.2 illustrate the expediency of using true anomaly as independent
variable. The plots present the integrand function \( U_T \cdot \cos(\nu \cdot M) \) as function of mean and true anomalies (dependence of mean anomaly – plot 1, dependence of true anomaly – plot 2).

For obtaining spline-polynomial of true anomaly for \( I_{\nu}(g) \) the substitution of the integration variable is used: \( dM = (1/\eta) \cdot (r/a)^2 \cdot df \). Using the calculated values of the integrand function \( g \cdot \exp(-\sqrt{1-\nu \cdot M}) \) the interpolation spline-polynomial is constructed for the uniform net. Integrating this polynomial we obtain spline-polynomial representing the function \( I_{\nu}(g) \). The program developed on the basis of suggested algorithm uses not more than 16 interpolation nodes.

The calculations of the short-periodic perturbations from the Tesseral harmonics of geopotential use the partial derivatives of the functions \( V_T^q \) (see (5.2)) with respect to orbital elements. For Kepler’s, Delaunay’s and Lagrange’s SESs these partial derivatives satisfy the equations, similar to (5.2) and the described above technique is used for computation. The exceptions are: the partial derivatives in semi-major axis \( a \) for Kepler’s SES and with respect to energy parameter \( L = (\mu \cdot a)^{1/2} \) in Delaunay and Lagrange systems. The expressions for these derivatives can be obtained by differentiating (with respect to \( a \) or \( L \)) the expressions, giving the solution to (5.2). The relationships \( \partial F/\partial L = \partial F/\partial \nu \cdot \partial \nu /\partial L, \partial F/\partial a = \partial F/\partial \nu \cdot \partial \nu /\partial a \) \( (\partial \nu /\partial L = (3/L) \cdot \nu, \partial \nu /\partial a = (3/2) \cdot \nu/a) \) should be used in this process.

The operator of the secondary averaging \( I^* \) for the Tesseral harmonics is constructed in the following way:

1. Two resonance zones are determined \( |\nu - p| < \alpha_1 \) \( \quad \alpha_1 \leq |\nu - p| < \alpha_2 \) \( (p – the \) entire, closest to \( \nu \)).
2. When \( \nu \) falls into the first zone (of deep resonance) the operator of secondary averaging \( I^* \) is set as identical. The right sides of DADEM include the resonance terms and the long-periodical perturbations are equal to zero.
3. When \( \nu \) falls into the second zone (of boundary resonance) the operator of secondary averaging \( I^* \) is set to zero. Long periodical perturbations are determined from resonance terms. In this case the denominators \( p \cdot n - q \cdot n^{'} \) are not too small.
4. When \( \nu \) is outside these zones there is no resonance and thus the operator of the first averaging \( I \) is zero. In this case nothing enter the right sides of DADEM. The long periodicals are zero as well.

In the algorithm the resonance terms are calculated on the basis of classical expansion of the geopotential into trigonometric series in the multiples of mean anomaly [11]. There are not many resonance terms \( (each \ index \ q \ correspond \ to \ not \ more \ than \ one \ p, \ which \ correspond \ to \ the \ resonance \ (see \ (5.1)) \), thus their computation does not consume a lot of time.

\section*{§ 6 Gravitational perturbations from the Moon.}

The algorithm calculates three types of gravitational perturbations from the Moon:

1. Short periodicals of the first order.
2. Long periodicals.
3. The right sides of DADEM.

Calculation of the short periodicals is connected with the solution of the equations (2.5). Lidov’s technique [13] (see also § 8) is convenient for this task. This method is applicable for solving the equations of the form:

\[ n \cdot \partial W/\partial M + \partial W/\partial t = F(M, t), \quad (6.1) \]

where the function \( F(M, t) \) slowly changes with time. The solution is represented in the form of a series (see Appendix):
\[ W = \sum_m (-1)^m \cdot \partial^m \frac{\Psi^{m+1}(F)}{\partial^m t} , \ m = 0 \div \infty , \tag{6.2} \]

where \( \Psi(F) = \Phi(F) - \int_0^M \Phi(F) \, dM / (2\cdot\pi) , \)

\[ \Phi(F) = \int_0^M F(\xi,t) \, d\xi / n , \]

Let us remind the known expansions, used in satellite motion theory.
Denote \( U^B \) – the force function of the perturbing body. According to (3.3):

\[ U^B = \mu^B \cdot \left( 1/\Delta^B - (r \cdot r_B) / r_B^3 \right) \tag{6.3} \]

Expand the right side of (6.3) into a series in Legendre polynomials:

\[ U^B = (\mu^B / r_B) \cdot \sum_{m=2}^{\infty} \left( r / r_B \right)^m P_m(\cos(\gamma)) , \tag{6.4} \]

where \( \gamma \)- the angle between the vectors \( r \) and \( r_B \).

The expansion into associated Legendre functions can be obtained from (6.4) using composition theorem for Legendre polynomials:

\[ U^B = (\mu^B / r_B) \cdot \sum_{n=2}^{\infty} \sum_{q=0}^{n} c_{nq} \cdot P_n^{(q)}(sin(\delta)) \cdot P_n^{(q)}(sin(\delta^B)) \cdot \cos(q \cdot (\alpha - \alpha^B)) , \tag{6.5} \]

\( \alpha, \delta \) - declination and right ascension of the point,
\( \alpha^B, \delta^B \) – declination and right ascension of the perturbing body,
\( c_{nq} = 2 \cdot \delta^q \cdot (n-q)! / (n+q)! \),
\( \delta^q = 1/2 \) for \( q=0 \) and \( 1 \) for \( q \neq 0 \).

Let us consider the orbital coordinate frame \( (OCF) \) with the following axes:
1. \( X' \) axis is directed towards the ascending node of the orbit,
2. \( Z' \) axis – is normal to the orbital plane \( (normalized \ vector \ product \ of \ the \ radius-vector \ and \ velocity) \),
3. \( Y' \) axis completes the system (to the right-hand one).

The transformation of coordinates from GUCF to OCF has the form:

\[ x' = x \cdot \cos(\Omega) + y \cdot \sin(\Omega) , \tag{6.6} \]
\[ y' = -x \cdot \cos(i) \cdot \sin(\Omega) + y \cdot \cos(i) \cdot \cos(\Omega) + z \cdot \sin(i) , \]
\[ z' = x \cdot \sin(i) \cdot \sin(\Omega) - y \cdot \sin(i) \cdot \cos(\Omega) + z \cdot \cos(i) . \]

The expression (6.5) has the following form in OCF:
\[ U^B = \left( \frac{\mu^B}{r^B} \right) \sum_{n=2}^{\infty} (r / r^B)^n \cdot \sum_{q=0}^{n} c'_{nq} \cdot P_n^{(q)}(\sin(\delta^B)) \cdot \cos(q \cdot (u - \alpha^B)) , \quad (6.7) \]

\[ n = 2 + \infty , \quad q = 0 + n , \quad n - q - \text{even} , \]
\[ \alpha^B , \quad \delta^B - \text{spherical angles of the perturbing body in OCF} , \]
\[ u - \text{latitude argument of the satellite} , \]
\[ c'_{nq} = (-1)^{n+q/2} \cdot \delta^B(n-q)! / (2^{n-1}) \cdot ((n-q)/2)! \cdot ((n+q)/2)! \]

We will write the expansion (6.7) in the form:

\[ U^B = \Re \left( Z^B \right) , \quad (6.8) \]

\[ Z^B = \left( \frac{\mu^B}{a^B} \right) \sum_{n=2}^{\infty} \sum_{q=0}^{n} Y^B_{nq} , \quad n - q - \text{even} , \]
\[ Y^B_{nq} = (a/a^B)^n \cdot Z^B_{nq} , \]
\[ Z^B_{nq} = \Phi_{nq}(p_B, e_x^B, e_y^B, e_z^B) \cdot G_{nq}(e, \omega; E) , \quad (6.9) \]

where \( \Phi_{nq}(p_B, e_x^B, e_y^B, e_z^B) = c'_{nq} \cdot (a^B/a)^{(n+1)} \cdot S_{nq}(e_x^B, e_y^B, e_z^B) - \quad (6.10) \)

\( \Phi_{nq}(p_B, e_x^B, e_y^B, e_z^B) \) - polynomial of \( p_B, e_x^B, e_y^B, e_z^B \),
\( p_B = a^B/r^B \), \( e_x^B, e_y^B, e_z^B \) - coordinates of the normalized radius-vector of the perturbing body in OCF,

\[ S_{nq}(e_x^B, e_y^B, e_z^B) = P_n^{(q)}(\sin(\delta^B)) \cdot \exp(-\sqrt{-1} \cdot q \cdot \alpha^B) - \quad (6.11) \]

Spherical function (uniform polynomial of \( n \)-th order in \( e_x^B, e_y^B, e_z^B \), satisfying Laplace equation),

\[ G_{nq}(e, \omega; E) = (r/a)^n \cdot \exp((-1)^{1/2} \cdot q \cdot f) \cdot \exp(\sqrt{-1} \cdot q \cdot \omega) , \quad (6.12) \]

\( E - \text{eccentric anomaly} \).

Let us expand \( G_{nq} \) into trigonometric series in multiples of eccentric anomaly. We will use known formulas:

\( (r/a) = 1 - e \cdot \cos(E) \), \( (r/a) \cdot \cos(f) = \cos(E) - e \), \( (r/a) \cdot \sin(f) = \eta \cdot \sin(E) \) (\( \eta = \sqrt{1 - e^2} \)).

\[ G_{nq}(e, \omega; E) = \sum_{p=-n}^{n} Y_{nq}^P \cdot \exp(\sqrt{-1} \cdot (p \cdot E + q \cdot \omega)) , \quad (6.13) \]

where \( Y_{nq}^P \) - eccentricity functions (polynomials in \( e \) and \( \eta \)).

The formula (6.13) represents the functions \( G_{nq} \) in finite form.

Consider the algorithm for solving the equation:

\[ n \cdot \partial V^B / \partial M + \partial V^B / \partial t = -(Z^B - I(Z^B)) , \quad (6.14) \]

where \( I - \text{averaging operator} : I(Z^B) = \int_{0}^{2 \cdot \pi} Z^B(M, t) \ dM / (2 \cdot \pi) \)
Due to linearity of the equation (6.14) and formula (6.8) the solution can be represented in the form:

\[ V^\beta = - \left( \mu^\beta / a^\beta \right) \cdot \sum_{n=2}^{\infty} \sum_{q=0}^{n} \left( a / a^B \right)^n \cdot \Psi_{nq}^\beta \cdot n - q \text{ - even}, \tag{6.15} \]

where \( \Psi_{nq}^\beta \) - solutions of the equations:

\[ n \cdot \partial \Psi_{nq}^\beta / \partial M + \partial \Psi_{nq}^\beta / \partial t = Z_{nq}^\beta - I(Z_{nq}^\beta). \tag{6.16} \]

For solving equations (6.16) we use Lidov’s method and representation (6.9) of the functions \( Z_{nq}^\beta \):

\[ \Psi_{nq}^\beta = \sum_{m} (-1)^m \cdot \Psi^{(m+1)} (G_{nq} - I(G_{nq})) \cdot \partial^m \Phi_{nq} / \partial t, \quad m = 0 + \infty \tag{6.17} \]

1. **Recurrent determination of \( \Psi^m (G_{nq} - I(G_{nq})) \)**

Let us replace in (54) the integration over mean anomaly by the integration over eccentric anomaly \( dM = (r/a) dE \). Then use the expansion (6.13).

\[ \Psi^0 (G_{nq} - I(G_{nq})) = G_{nq} - I(G_{nq}) = \sum_{p=-n}^{n} \gamma_{nq}^0 p \cdot \exp \left( \sqrt{-1} \cdot (p \cdot E + q \cdot \omega) \right), \tag{6.18} \]

\[ \Psi^{(m+1)} (G_{nq} - I(G_{nq})) = \left( \sqrt{-1} / n \right)^{(m+1)} \cdot \sum_{p=-(n+m+1)}^{-(n+m+1)} \gamma_{nq}^{(m+1)} p \cdot \exp \left( \sqrt{-1} \cdot (p \cdot E + q \cdot \omega) \right), \tag{6.19} \]

\[ \gamma_{nq}^{(m+1)} p = \left( \gamma_{nq}^m p - (e/2) \cdot (\gamma_{nq}^m (p+1) + \gamma_{nq}^m (p+1)) / p \right) - \text{for } n+m > |p| > 0, \]

\[ \gamma_{nq}^{(m+1)} p = \left( \gamma_{nq}^m p - (e/2) \cdot \gamma_{nq}^m (p-1) / p \right) - \text{for } p = n+m, \]

\[ \gamma_{nq}^{(m+1)} p = - \left( e/2 \right) \cdot \gamma_{nq}^m (p-1) / p - \text{for } p = n+m+1, \]

\[ \gamma_{nq}^{(m+1)} p = \left( \gamma_{nq}^m p - (e/2) \cdot \gamma_{nq}^m (p-1) / p \right) - \text{for } p = (n+m), \]

\[ \gamma_{nq}^{(m+1)} p = - \left( e/2 \right) \cdot \gamma_{nq}^m (p-1) / p - \text{for } p = -(n+m+1), \]

\[ \gamma_{nq}^{(m+1)} p = \left( e/2 \right) \cdot \left( \gamma_{nq}^{m+1} (p-1) + \gamma_{nq}^{m+1} (p+1) \right) - \text{for } p = 0. \]

2. **Recurrent determination of \( \partial^m \Phi_{nq} / \partial t \)**

Let us temporary denote \( F^{(m)} = \partial^m F / \partial t \). Note that\(^6\)

\[ \Phi_{nq}^{(m)} = \Phi_{nq} (p_B^{(k)}, e_x^{(k)}, e_y^{(k)}, e_z^{(k)} ; k=0,1,\ldots,m) \tag{6.20} \]

\[ \Phi_{nq}^{(m+1)} = \sum_{k=0}^{m} \left( (\partial \Phi_{nq}^{(m)} / \partial p_B^{(k)}) p_B^{(k+1)} + (\partial \Phi_{nq}^{(m)} / \partial e_x^{(k)}) e_x^{(k+1)} + (\partial \Phi_{nq}^{(m)} / \partial e_y^{(k)}) e_y^{(k+1)} + (\partial \Phi_{nq}^{(m)} / \partial e_z^{(k)}) e_z^{(k+1)} \right), \]

\( p_B^{(k)} \) - does not depend on satellite orbit,

\( e_x^{(k)} = e_x^{(k)} \cdot \cos(\Omega) + e_y^{(k)} \cdot \sin(\Omega) \),

\( \Phi_{nq}^{(m)} \) is omitted.

\(^5\) Further not to be confused with index of power!

\(^6\) The upper index of the perturbing body \( B \) is omitted.
\[ e_{y}^{(k)} = -e_{z}^{(k)} \cos(i) \cdot \sin(\Omega) + e_{r}^{(k)} \cos(i) \cdot \cos(\Omega) + e_{z}^{(k)} \cdot \sin(i), \]
\[ e_{z}^{(k)} = e_{r}^{(k)} \sin(i) \cdot \sin(\Omega) - e_{y}^{(k)} \cdot \sin(i) \cdot \cos(\Omega) + e_{z}^{(k)} \cdot \sin(i). \]

The algorithm uses spline-polynomials for representation of \( p_{B}, e_{x}^{B}, e_{y}^{B}, e_{z}^{B} \). Coefficients of these polynomials are calculated using the values, found according to Hill-Brawn formulas.

The formulas (6.15) \((6.20)\) give the completed solution of the equation (6.14). The solution of the equation (2.5) of the first order is connected with the solution of (6.14) by the relationship:
\[ W_{1} = W_{1}^{B} = \Re(\mathbf{I}^{B}). \]

The algorithm takes into account the short periodic Lunar and Solar perturbations of the first order. For satellites with orbital periods \( \leq 1 \text{ day} \) this is quite enough.

The formulas for short periodic perturbations of the first order from the Moon and the Sun can be obtained from (6.15) \((6.20)\). (For the Moon, when we use Lidov’s technique (see (6.17)), \( m \leq 2 \) is sufficient, and for the Sun \( m \leq 1 \).

The described technique for the calculation of Lunar and Solar short periodicals does not include limitations for the eccentricity. The calculation formulas use finite functions of \( e \) and \( \eta \).

Let us consider the technique for determination of the doubly averaged Hamilton’s function and the long-periodic perturbations for the perturbing body.

First we will obtain formulas for singly averaged Hamiltonian of the perturbing body.

According to the introduced denotation:
\[ H^{*}_{1}^{B} = -I(U^{B}) = -\Re(\mathbf{I}(Z^{B})) \], where \( I \) – averaging operator.

To obtain \( I(Z^{B}) \) we will return to the expansion (6.5) of the function \( U^{B} \). Let the reference coordinate frame in (6.5) be GUCF. We will use the known expansion [11]:
\[ P_{n}(q) (\cos(\delta) \exp((-1)^{1/2}q \cdot \alpha)) = \sum_{k=-n}^{n} \left( -1 \right)^{k \cdot \frac{n-q}{2}} \cdot A_{nq}^{k}(q) \exp(\sqrt{-1} \cdot k \cdot u), \quad (6.21) \]
\[ n - k - \text{ even}, \]
\[ \alpha, \delta - \text{declination and right ascension of the point in GUCF,} \]
\[ u = f + \omega - \text{latitude argument,} i - \text{inclination,} A_{nq}^{k}(i) - \text{inclination functions.} \]

Then we can obtain the complex form \( Z^{B} \) of function \( U^{B} \) from (6.5) and (6.21):
\[ Z^{B} = (\mu^{B}/a^{B}) \cdot \sum_{n=2}^{\infty} (a/a^{B})^{n} \cdot \sum_{q=0}^{n} \Phi_{nq}^{B} (-1)^{(n-q)/2} \cdot \sum_{k=-n}^{n} A_{nq}^{k}(i) \left[ (r/a)^{n} \exp(\sqrt{-1} \cdot (k \cdot f)) \right] \cdot \exp(\sqrt{-1} \cdot (k \cdot \omega + q \cdot \Omega)), \quad (6.22) \]
\[ n - k - \text{ even}. \]
\[ \Phi_{nq}^{B} = c_{nq}^{*}(a^{B}/r_{B})^{(n+1)} \cdot P_{n+1}^{(n+1)}(\sin(\delta^{B})) \cdot \exp(-\sqrt{-1} \cdot q \cdot \alpha^{B}) = c_{nq}^{*} \cdot P_{n+1}^{(n+1)} \cdot S_{nq}(e_{x}^{B}, e_{y}^{B}, e_{z}^{B}), \]

According to the definition of the averaging operator (see (6.14)), (6.22) yields:

\[ Eccentricity \text{ functions } Y_{nq}^{B} - \text{polynomials in eccentricity } e. \]
\begin{equation}
Z^B* = I(Z^B) = (\mu^B/a^B) \cdot \sum_{n=2}^{\infty} (a/a^B)^n \cdot \sum_{q=0}^{n} \Phi^B_{nq} \cdot (-1)^{(n-q)/2} \cdot \sum_{k=-n}^{n} A^n_{kq} i \cdot Y_{(n+1),q,0} \cdot \exp(-1 \cdot (k \cdot \omega + q \cdot \Omega)) \ , \ n-k \text{ odd} \ .
\end{equation}

The smoothing in time is performed to obtain doubly averaged Hamiltonian.

\begin{equation}
H^* _1 = - I^* (I(I(U^B))) = - \Re \Phi^* (Z^B) .
\end{equation}

\begin{equation}
I^* (Z^B) = (\mu^B/a^B) \cdot \sum_{n=2}^{\infty} (a/a^B)^n \cdot \sum_{q=0}^{n} \Phi^B_{nq} \cdot (-1)^{(n-q)/2} \cdot \sum_{k=-n}^{n} A^n_{kq} i \cdot Y_{(n+1),q,0} \cdot \exp(-1 \cdot (k \cdot \omega + q \cdot \Omega)) \ , \ n-k \text{ odd} .
\end{equation}

where \( \Phi^B_{nq} = I^* (\Phi^B_{nq}) \). 

The last equality is obtained due to linearity of smoothing operation and is produced from (6.23) by replacement of \( \Phi^B_{nq} \) by \( \Phi^B_{nq}^* \).

Determination of the long periodic perturbations for the Moon and the Sun is reduced to solution of simplified equations (2.5):

\begin{equation}
\frac{\partial W^*}{\partial t} = F , \quad (6.25)
\end{equation}

since the right sides of these equations do not depend on “fast” angle \( M \). Solution of the equation (6.25) is reduced to integration over time:

\begin{equation}
W^* = \int F \, dt . \quad (6.26)
\end{equation}

Let us consider the technique of smoothing and determination of the long periodic perturbations, implemented in the algorithm.

Let \( F \) be a function of time. We take a long time interval (several tens of periods of the perturbing body) \([t_0, t_1]\) and divide it into medium intervals (several periods of the perturbing body). Each medium interval we will divide into small intervals ( \( \sim 0.01 \text{ to } 0.1 \text{ of the period of the perturbing body} \)). Using the generated partition of the long interval into small ones we shall construct the spline-polynomial (\( \sim 4^{th} \text{ power} \)) \( F_s \) for the smoothed function \( F \). We consider \( F_s \) a good approximation for \( F = F_s \). Among all the spline-polynomials of the given power (normally \( 2 \text{ to } 4 \)), defined by the partition into medium intervals, we will find \( F^* \), closest to \( F_s \) regarding the metric\( 8 L^2 \). Determination of \( F^* \) is reduced to solution of the system of linear equations. We define the smoothing operator \( I^* : I^* (F) = F^* \). Introducing denotation\( ^* F = \int F = t_0 \int^t (F - F^*) d\xi \) the long periodic perturbations of the first order are determined by the function:

\begin{equation}
V^* _1 = - \int (Z^* ) = - (\mu^B/a^B) \cdot \sum_{n=2}^{\infty} (a/a^B)^n \cdot \sum_{q=0}^{n} f(\Phi^B_{nq}) . \quad (6.27)
\end{equation}

\[8 \text{ In } L^2 \text{ metric the distance between two functions is defined as the square root from the integral of the squared difference between them.}\]

\[9 \text{ } F^* \text{ is represented by spline-polynomial within the interval } [t_0, t_1].\]
\[-1\]^{(n+q)/2} \cdot \sum_{k=-n}^{n} A_{nq}^k (i) \cdot Y_{(n+1),q}^0 \cdot \exp(\sqrt{-1} \cdot (k \cdot \omega + q \cdot \Omega)) \cdot n - k \text{ even}.

\[ W^*_1 = \Re \varepsilon (V^*_1 B). \]

The coefficients of spline-polynomials \( I^*(\Phi^B_{nq}) \) and \( \tilde{I}^*(\Phi^B_{nq}) \) used in the algorithm, are determined once and then are used for all propagated satellites.

For the Moon we take into account the long periodic perturbations of the second order, accounting of the cross influence of Moon gravity and the second zonal harmonic of the geopotential. The following equation (see (2.5)) must be solved to determine them:

\[
\partial W^*_2 / \partial t = 2 \cdot ( ( [H^*_1, W^*_1] + [H^{**}_1, W^*_1] ) - H^{**}_2 ) \quad (6.28)
\]

Select the largest terms in \( H^{**}, H^*_1 \) and \( W^*_1 \):

\[ H^*_1 \approx H^*_1 Z^2, \quad H^{**}_1 \approx H^{**}_1 Z^2, \quad W^*_1 \approx W^*_1 B. \quad (6.29) \]

When this approximation is used we can set \( H^{**}_2 = 0 \). After that the equation (6.28) will take the form:

\[
\partial W^*_2 / \partial t = [H^*_1 Z^2, W^*_1 B] \quad (6.30)
\]

Using the representation of Poisson brackets in Kepler’s SES (2.2), we have:

\[
\partial W^*_2 / \partial t = k_2' \cdot \partial H^*_1 Z^2 / \partial \cos(i) \cdot ( \cos(i) \cdot \partial W^*_1 B / \partial \omega - \partial W^*_1 B / \partial \Omega ) + k_1' \cdot \partial H^*_1 Z^2 / \partial \varepsilon \cdot \partial W^*_1 B / \partial \omega , \quad (6.31)
\]

where \( k_2' = 1 / (L \cdot \eta) \), \( k_1' = \eta / L \).

Integration of the right side of the last equation is sufficient to obtain its solution. Technically this means the calculation of integrals over time:

\[
\int_{t_0}^{t} [I^* (\Phi^B_{nq})] (\xi) \, d\xi
\]

In practice, all the things needed for calculation of the long periodicals, determined by the function \( W^*_2 \), are calculated in the course of computation of the long periodicals of the first order. Thus the computation can be arranged very economically.

§ 7 Gravitational perturbations from the Sun.

In section § 8 we discuss the rationale for rejecting the second averaging for the gravitational perturbations from the Sun.

For the Sun we calculate only the short periodic perturbations of the first order and the right sides of DADEM. The operator of the secondary averaging \( I^* \) is considered identical. Hence, the long periodicals are equal to zero.
The technique for determination of the short periodic perturbations was discussed above (formulas (6.8) + (6.20)). Singly averaged Hamiltonian is obtained using the formulas (6.8) + (6.14):

\[ H^S_{1_S} = I(H^S_{1_S}) = - I(U^S_{1_S}) = - \Re \varepsilon (I(Z^S)) , \]

\[ I(Z^S) = \int \bar{Z}^S (M, t) dM / (2 \pi) , \]

\[ I(Z^S) = (\mu^S / a^S) \sum_{n=2}^{\infty} \sum_{q=0}^{n} I(Y^S_{nq}) , \quad n - q \text{ even} , \]

\[ I(Y^S_{nq}) = (a / a^S)^n I(Z^S_{nq}) , \]

\[ I(Z^S_{nq}) = \Phi_{nq}(p_B, e_x^S, e_y^S, e_z^S) \cdot I(G_{nq}(e, \omega; E)) , \]

\[ I(G_{nq}(e, \omega; E)) = Y_{(n+1), q}^{0} \cdot \exp(\sqrt{-1} q \omega) \]

Finally we have:

\[ H^S_{1_S} = - (\mu^S / a^S) \sum_{n=2}^{\infty} \sum_{q=0}^{n} (a / a^S)^n \Re \varepsilon (\Phi_{nq}(p_B, e_x^S, e_y^S, e_z^S) Y_{(n+1), q}^{0} \exp(\sqrt{-1} q \omega)) , \]

\[ n - q \text{ even} , \]

\[ (7.1) \]

**§ 8 Brief history of algorithm development**

Several years have passed from the beginning of the development of the procedure until the efficient performance of the program code was attained. In the course of this period the procedure have been changed and updated many times. The first program codes turned to be not enough accurate and fast. The search for the route causes of this effects and the relevant solutions of the raising difficulties have resulted in a whole set of technical solutions, which form the basis of the algorithm. We would like to speak about these solutions and the ways, which led to them.

First we decided to take the perturbation theory for the basis of the algorithm. We think that the other techniques (e.g. revolution-by-revolution integration) are either not accurate enough or can be reduced to perturbation theory.

The force function corresponding to gravitational potential of the homogeneous sphere was taken for the unperturbed force function. This is traditional for the semi-analytical algorithms. However, certain techniques exist that use the function of the problem of the gravity of two unmovable centers (spaced for imaginary distance) for the unperturbed force function. [9]. This allows to account of the perturbations from C20 just in the unperturbed motion. This technique can be efficient when the perturbations from C20 are significantly greater compared to all the others (we can reduce the order of the used expansions in terms of small parameter of the perturbation theory). In other cases it does not provide any advantages and using it we would only accomplish the computation scheme (because of complex SES used). For the satellites with high eccentricities the gravitational perturbations from the Moon and the Sun can exceed the perturbation from C20 thus the application of the technique [9] is not justified for this case.

\[ ^{10} \text{The notation correspond to the formulas (6.8) + (6.14) where the upper index }^B \text{ is replaced by }^S. \]
The semi-analytical algorithms, known to us (having the required prediction accuracy) used the averaging only along satellite orbit. This does not allow to make the step of the integration of the averaged equations wide enough, limiting this step by approximately 1 day. The route cause is the gravitational perturbations from the Moon, which have the orbital period \( \sim 28 \) days. Since we aspired to design a fast procedure, we decided to introduce the averaging over the motion of the perturbing body (the Moon, the Sun) together with the averaging along satellite orbit right from the very beginning.

The first code implementing this averaging demonstrated poor accuracy. This was confusing, because the same averaging, implemented in GEO propagator provided good results. The procedures were different in representation of the perturbations and were used for different classes of satellites. The experimental investigation revealed the following:

1. Prediction accuracy decreases abruptly with the increase of orbital eccentricity.
2. Application of the model, including the Earth gravitational perturbations only, provides very accurate prediction.
3. Application of the model, including only the gravity of the perturbing body, reduces the prediction errors significantly.

It became clear that the account of the cross influence of the perturbations from the Earth and the perturbing body is needed. Thus we must include the terms of not less than second order in the used perturbation theory expansions. On the other hand the averaging only along the orbit of the satellite with the first order of accuracy provided good results (leading, however, to significant reduction of the step of the integration of the averaged equations). Thus the conclusion was that the chosen averaging technique and high orbital eccentricity generated the issue.

The first impression was to leave the idea of averaging over the motion of the perturbing body and may be to drop the idea of creating fast procedure at all. Nevertheless, in certain time the compact formulas, taking into account the cross influence of C20 and the perturbing body gravity have been obtained. This became possible due to the fact, that the cross influence manifests itself most of all in the averaging over the motion of the perturbing body and this averaging is more simple, than the averaging along satellite orbit.

For the Moon the cross influence with the second zonal harmonic is accounted of only in the junior term of the second order. In accounting of the perturbations from the Sun, the account of the cross influence of C20 with the terms up to the 4\(^{th}\) order, did not yield the required prediction accuracy. We did not perform detailed investigation of this fact (may be the calculation formulas were not free from mistakes). The Sun has rather long period of motion with respect to the Earth = 1 year, thus the averaging of the respective perturbations only along satellite orbit in practice does not lead to the increase of the step of integration of the doubly averaged equations of motion.

In addition to the account of the cross influence of C20 and the gravity of the perturbing body, important modification was made in the averaging technique. The earlier versions of the procedure performed one common averaging in both motions – along satellite orbit and the motion of the perturbing body. We divided this averaging into two independent ones. First the averaging along satellite orbit is performed. This yields the averaged equation of motion and the short periodic perturbations. Then the averaged equation of motion is taken as initial and the secondary averaging is performed to reduce the time-dependence of the right sides. This yields the doubly averaged equation of motion and the long periodicals. The general solution is the combination of the solution of the doubly averaged equation (determined numerically with large step (\( \sim 10 \div 20 \) days)), long periodic perturbations and short periodic perturbations. The short periodicals are calculated in the point, corrected by long periodicals – this is the key element of the technique. Thus we can reduce the order of the perturbation theory expansions involved. We should note that the two-stage averaging
scheme is useful right for the satellites with high eccentricities. Our prediction procedure for GEO uses one stage averaging scheme that gives satisfactory accuracy for eccentricities \( \sim 0.1 \).

For the orbits with high eccentricities the account of the perturbations from Tesseral harmonics of geopotential is complicated mainly by the need to include the tremendous amount of the terms of the expansions, arriving in the solution of the problem. We tried to get round this problem, reducing it to numerical computation of the coefficients of the spline-polynomials. Notwithstanding the sophistication of this technique, it was efficient enough and had not changed since the time of its practical implementation. As it turned out, only 8-16 interpolation nodes in the circle are sufficient. The account of the resonances – is one more problem, related to Tesseral harmonics. When we have the resonance the resonance terms must be incorporated into the averaged Hamilton’s function. The choice of the threshold for the difference in frequencies, used for attributing the respective term to the resonance ones is the issue. In case this threshold is low, then at the boundary the short periodicals become too great and this is not acceptable. If the threshold is rather high, then we have to integrate the averaged equations with large step. We reduced the acuteness of this problem by introducing the intermediate resonance zone. When the resonance terms fall into it they are considered long periodic and are not incorporated into the doubly averaged equations of motion.

After the discussed changes the program code implementing the algorithm achieved acceptable prediction accuracy, but the computation rate was below the expectations. Within a month interval our procedure was several times faster than the numerical one, but our aspiration was to change this ratio by the order of magnitude. We have done it. For this purpose we used different expansion of the perturbing functions of the Moon and the Sun. The classical expansion in GUCF, separating the orbit and the perturbing body and using the inclination functions was inefficient. Its application leads to bulky computations, and this was not obvious in the beginning. The expansion of the perturbing function in the orbital coordinate frame only partially separates the orbit and the perturbing body. The coordinates of the perturbing body are “mixed” with the orbital parameters, defining orbital plane. The orbital parameters, defining its geometry and the position of the point in orbit enter separate functions. The expansion itself becomes more compact and provides the possibility to obtain all the calculation formulas. Such an expansion was previously used by other authors. After its implementation the computation rate of our procedure increased approximately five times!

The development of the algorithm included determination of the ways to enhance performance of the code itself. One of these ways – arrangement of on-going computation mode (OCM). In this algorithm its implementation is much more simple than in the GEO propagator. \(^{11}\)

The OCM is not used in the calculation of the right sides of DADEM. If the ES has slightly changed, it will be better to increase the integration step, than to arrange OCM.

In the integration of the DADEM the arranging of the OCM means the remembering of the right sides and their epochs. If in the course of further calculations the epoch of prediction is between the epochs, recorded previously, no calculation of the right sides takes place and the recorded ones are used. This technique is used multiple propagation of one and the same satellite.

For the calculation of the short periodic and long periodic perturbations the OCM is organized as follows:

1. Doubly and singly averaged orbital elements (excluding the “fast” angle) for which short and long periodicals are calculated, are remembered along with the respective amplitudes of the perturbations.

\(^{11}\) The algorithm for GEO satellites uses three-stage calculation scheme. This one uses one-stage scheme.
2. Prior to the calculation of perturbation amplitudes, the closeness of current slow ES to the remembered ones is checked. If they are close, the amplitudes are not calculated and the remembered ones are used. If they differ significantly, then the remembered amplitudes are calculated and the respective ES are updated as well.

The tests of the algorithm showed, that for successive propagation of one and the same satellite the described OCM allows to avoid calculation of the amplitudes of short and long periodicals within time intervals of several days. This provides significant economy for the general scheme of data processing.

The efficient design of the computation scheme was an issue as well. The algorithm uses numerous special functions and their amount is determined by the set of parameters, defining the perturbing factors, accounted of. Application of general computation procedures for the calculation of special functions takes much more time than employment of direct formulas. To overcome this problem we have written the generators of the calculation blocks. These programs, having the input of the involved parameters, produce the texts of the codes, calculating directly the variables, used in the program. The produced texts are used by prediction program. In this way several blocks for the calculation of the inclination and eccentricity functions have been generated. When we change the parameters of the perturbations’ model of the propagator, we must use the generators of the calculation blocks and then recompile the program. This approach appears to be very useful, but the subject is not in the scope of the present paper.

§ 9 Appendix

9.1. Fundamentals of Hori – Deprit method

From the point of view of an engineer the perturbation theory is a set of general formulas, allowing the construction of calculation program. However, the clear understanding of the basic ideas constituting the basis of this theory provides the good foundation for reasonable handling the formulas. The laconic elegance of the representation of the Hori – Deprit technique is rather attractive. At the first glance the resulting formulas look too abstract for using in applied programs. In fact the situation is different. The formulas of Hori – Deprit method can be easily displayed in the chosen coordinate frame (including non-canonical ones) yielding the calculation formulas for the applied problem.

Basic notions and terms

We will use the following notions of differential geometry:
1. Differentiable manifold. Will be denoted $M$ ,
2. Tangent fibration of the manifold $M$. Will be denoted $TM$ ,
3. Cotangent fibration of the manifold $M$. Will be denoted $T^*M$ ,

Motion determined by the vector field $h : M \rightarrow TM$ :
Mapping $g$ of the interval of time axis $I = (a,b)$ into manifold $M$ -
$g : I \rightarrow M$ , satisfying the differential equation $\frac{dx}{dt} = h(x)$ ,
where $x \in M$, $p^\circ h = E$, $p : TM \rightarrow M$ – natural projection , $E : M \rightarrow M$, $E(x) = x$ – identical motion.

Phase flow, determined by the vector field $h : M \rightarrow TM$ :
Mapping $G : M \times I \rightarrow M$ satisfying the conditions:
1. $G(\bullet, t_0) = E$ for certain fixed $t_0 \in I$.
2. $G(x_0, \bullet) = motion$, determined by the vector field $h$, for arbitrary fixed $x_0 \in M$.

Symplectic structure –
1. Differentiable manifold $M$.
2. Non-singular, closed differential 2-form $\omega$ on $M$. 

20
Hamiltonian system (autonomous Hamiltonian system) – symplectic structure and function $H$ on the manifold $M$.

Hamilton’s function (Hamiltonian) – function $H$.

Non-singularity of the form $\omega$ allows natural introduction of the biunivocal mapping $D$:

$$
D : T^*M \rightarrow TM
$$

$$
\eta^*(\xi) = \omega(\eta, \xi) ; \eta, \xi \in T^*M ; \eta = D(\eta^*) ;
$$

(9.1)

$$
p^*(\eta) = p(\eta) = p(\xi) ;
$$


The mapping $D$ allows to establish correspondence between a differentiable function $A$ on $M$ and a vector field. Let $A \in C^1(M)$ be a smooth function on the manifold $M$, its differential $dA$ – covector field on $M$, that will be denoted $a^*$. The mapping $D$ allows to obtain vector field $a = D(a^*)$:

$$
dA(\xi) = \omega(a, \xi) ;
$$

(9.2)

for each $\xi \in TM$ that $p(\xi) = p(a) = p^*(dA)$.

The motion in the Hamiltonian system with the Hamilton’s function $H$ is determined by the vector field $h = D(dH)$.

Poisson brackets $[H_1, H_2]$ of two functions $H_1, H_2 \in C^1(M)$ are defined by vector fields $h_1 = D(H_1)$ and $h_2 = D(H_2)$:

$$
[H_1, H_2] = dH_1(h_2) = -dH_2(h_1) = \omega(h_1, h_2)
$$

(9.3)

As follows from the definition of Poisson brackets, they are anti-symmetrical and is the derivative of the first function with respect to the direction of the vector field, determined by the second one.

Thus follows that for the Hamiltonian system (with Hamilton’s function $H$) for any smooth function $A \in C^1(M)$ the following equality is valid:

$$
da/dt = [A, H]
$$

(9.4)

where $da/dt$ – time derivative along the motion.

Canonical mapping $\Omega : M \rightarrow M$ – retaining the form $\omega$:

$$
\omega(\xi, \eta) = \omega( d\Omega(\xi), d\Omega(\eta) )
$$

(9.5)

where $d\Omega$ - differential of the mapping $\Omega$.

$\xi, \eta \in TM$, $p(\xi) = p(\eta)$

As follows from the definition of canonical mapping, for the vector fields $h$ and $h_1$, determined by the functions $H$ and $H^* = H \circ \Omega$, the relationship $h = d\Omega( h^* )$ is valid. Really, $\omega(h^*, \eta) = dH^* (\eta) = dH(d\Omega(\eta)) = \omega(h, d\Omega(\eta))$, since $\eta$ is arbitrary, $p(\eta) = p(h^*) = p(h)$ and since $\Omega$ is canonical we find $h = d\Omega( h^* )$.

Thus follows, that canonical mapping $\Omega$ transforms the motion determined by Hamilton’s function $H_1$, into the motion, determined by Hamilton’s function $H$.

Non-autonomous Hamiltonian system – Hamiltonian system, where the Hamilton’s function is time-dependent: $H : MX \rightarrow \mathbb{R}$, where $\mathbb{R}$ – the set of real numbers.
Extending the dimension of the manifold we can make the transition from the non-autonomous Hamiltonian system to the autonomous one:

\[ M_a = M \times \mathbb{R}_t \times \mathbb{R}_T; \quad \omega_a(\xi, \eta)_t = \omega(\xi, \eta) + (T_t \xi - t_t \eta \cdot T_t); \quad H_a = H + T; \quad (9.6) \]

where \( \xi, \eta \in TM; \xi_a = (\xi, T_t \xi), T_T \xi, T_T \eta \in TM_a; \]

\[ p(\xi) = p(\eta); \quad p_a(\xi) = p_a(\eta), p_a: TM_a \to M_a \] – natural projection;

\( \mathbb{R}_t \) – one dimensional space of time \( t \),

\( \mathbb{R}_T \) – one dimensional space of additional parameter \( T \).

d\( H_t = (dH, \partial H / \partial t, 1) ; h_1 = (h, 1, -\partial H / \partial t) \)

Further we will consider autonomous Hamiltonian systems. Due to (6) this does not lead to limitations.

**Hori – Deprit method**

Let the Hamilton’s function depend on parameter \( \varepsilon \):

\[ H = H(\varepsilon), \quad \varepsilon \in I = [0, \alpha] \equiv H: M \times I \to \mathbb{R}. \]

The idea of the method is to find the canonical mappings \( \Omega(\varepsilon) \) of the manifold \( M \) into itself, that depend on parameter \( \varepsilon \):

\[ \Omega: M \times I \to M, \quad (9.7) \]

under fixed \( \varepsilon \in I \quad \Omega(\varepsilon) = \Omega(\bullet; \varepsilon) \) – canonical mapping.

Let us introduce the mapping:

\[ \theta: M \times I \to M \times I, \quad (9.8) \]

where \( \theta(\xi; \varepsilon) = (\Omega(\xi; \varepsilon), \varepsilon) \)

We will look for the mapping \( \Omega(\varepsilon) \) in the way, providing

\[ H^* = H \circ \theta, \quad (9.9) \]

where \( H^* = H^*(\varepsilon) \) satisfies the posed requirements, it is considered that \( H^*(\varepsilon = 0) = H \) satisfies these requirements. This allows taking identical motion \( E \) for \( \Omega(\varepsilon = 0) \)

The equality (9.9), due to canonical character of \( \Omega(\varepsilon) \), provides the transformation of motion in the Hamiltonian system with Hamilton’s function \( H^*(\varepsilon) \) into the motion in the Hamiltonian system with Hamilton’s function \( H(\varepsilon) \):

\[ m^* \to M \] – the motion, corresponding to \( H^*(\varepsilon) \)

\[ m = \Omega(\varepsilon)^* m^* : \mathbb{R} \to M \] – the motion, corresponding to \( H(\varepsilon) \).

The phase flows of Hamiltonian systems (autonomous and non-autonomous) are one-parametric families of canonical mappings.

The method of Hori - Deprit determines the mappings (9.7) as phase flow of the non-autonomous Hamiltonian system with Hamilton’s function \( W = W(\varepsilon) \) (\( \varepsilon \) plays the role of time in this system). The function \( W \) is called generator of mappings (9.7).

Having \( W \) determined, the mapping (9.7) can be found by expansion of the phase flow in the Hamiltonian system with function \( W \), in powers of \( \varepsilon \).

Take the derivative with respect to \( \varepsilon \) from the right side of (9.9). It is the derivative along the motion in the Hamiltonian system with Hamilton’s function \( W = W(\varepsilon) \), where \( \varepsilon \) plays the role of time. Operator of differentiation along the motion in this system will be denoted as \( D_W \). Then (9.4) and (9.9) yield:
\[ dH^\theta /d\varepsilon = (D_{W(H)}) \circ \theta = ( [H, W] + \partial H /\partial \varepsilon ) \circ \theta \quad (9.10) \]

Using (9.9) and (9.10) we can easily find higher derivatives:

\[ d^2H^\theta /d\varepsilon^2 = (D_{W^2(H)}) \circ \theta = \]

\[ \left( [[H,W],W] + 2([\partial H/\partial \varepsilon,W] + [H,\partial W/\partial \varepsilon] + \partial^2H /\partial \varepsilon^2) \right) \circ \theta \quad (9.11) \]

\[ d^3H^\theta /d\varepsilon^3 = (D_{W^3(H)}) \circ \theta = \]

\[ \left( [[[H,W],W],W] + 3\cdot[[\partial H/\partial \varepsilon,W],W] + 2\cdot[[H,\partial W/\partial \varepsilon],W] + [[H,W],\partial W/\partial \varepsilon] + 3\cdot[\partial^2H /\partial \varepsilon^2,W] + 3\cdot[\partial H/\partial \varepsilon,\partial W/\partial \varepsilon] + [H,\partial^2 W/\partial \varepsilon^2] + \partial^3 H /\partial \varepsilon^3 \right) \circ \theta , \]

\[ \ldots \]

Assuming that \( H, H^* \) and \( W \) are analytic functions of parameter \( \varepsilon \), we expand them into series:

\[ H(\varepsilon) = \sum\limits_n \varepsilon^n H_n, \quad H^*(\varepsilon) = \sum\limits_n \varepsilon^n H^*_n, \quad W(\varepsilon) = \sum\limits_n \varepsilon^n W_{(n+1)} ; \quad n=0+\infty \quad (9.12) \]

Using (9.10), (9.11), (9.12) we find:

\[ H^*_0 = H_0, \quad H^*_1 = H_1 + [H_0, W_1] \quad (9.13) \]

\[ 2H^*_2 = 2[H_2 + [H_0, W_2] + [H_1, W_1] + [H^*_1, W_1]], \]

\[ 6H^*_3 = 6[H_3 + 2[H_0, W_3] - [[H^*_1, W_1], W_1] + 4[H^*_2 - H_2], W_1] + [(H^*_1 - H_1), W_2] + 6[H_2, W_1] + 3[H, W_2] \quad (9.14) \]

The general technique for determination of the functions \( W_n \) is the successive solving of the partial derivatives equations:

\[ [H_0, W_j] = H^*_j - H_j + F_j, \quad \text{where} \quad F_j \text{ is determined via} \quad H_0, H_j, W_j, \quad j=1+ (i-1) \]

The functions \( W_j \) and \( H^*_j \), \( n=1,2... \) can be determined from equations (9.14) recurrently.\(^{12}\)

The family of mappings \( \Omega^{-1}(\varepsilon) \) is a phase flow, determined by the Hamilton’s function \(-W(\varepsilon)\).

Let us consider (on the manifold) the arbitrary function \( X \) \( \text{(which can be treated as coordinate function).} \)

Then:

\[ X \circ \Omega(\varepsilon) = \sum (\varepsilon^n /n!) \cdot d^n(X \circ \theta) /d\xi^n, \quad (9.15) \]

\[ X \circ \Omega^{-1}(\varepsilon) = \sum (\varepsilon^n /n!) \cdot d^n(X \circ \theta^{-1}) /d\xi^n, \]

where operators \( d^n(X \circ \theta) /d\xi^n \), \( d^n(X \circ \theta^{-1}) /d\xi^n \) use \( \xi = 0 \),

\[ \theta(x ; \xi) = (\Omega(x ; \xi) ; \xi) , \quad \theta^{-1}(x ; \xi) = (\Omega^{-1}(x ; \xi) ; \xi) , \quad n=0+\infty. \]

For the operators \( d^n(X \circ \theta) /d\xi^n \), \( d^n(X \circ \theta^{-1}) /d\xi^n \) we obtain formulas, similar to (9.10), (9.11). Taking the parameter in these formulas equal to zero and using (9.12), we get:

\(^{12}\) Partial derivatives equations (14) are linear. Consider the equations \([H_0, W] = A \cdot [H_0, W] = B \). Let \( W^A \) and \( W^B \) be the solutions of the equations. Then for the equation \([H_0, W] = a^*A + b^*B \) \( \text{where} \ a \text{ and} \ b \text{ are constants} \ \text{the function} \ W = a^* W^A + b^* W^B \) \( \text{is the solution} \text{(This feature is widely used for acquisition of short periodic and long periodic perturbations for Earth satellites).} \)}
Formulas (9.15), (9.16) can be used for representation of the mappings \( \Omega \) and \( \Omega^{-1} \) in coordinate form.

**Application to the problem of satellite motion.**

Let us consider the problem of material point motion in the force field. Denote \( x \) – vector of coordinates in inertial coordinate frame (\( GUCF \) for earth satellites), \( t \) - time. In this system the equation of motion has the form:

\[
d^2x/dt^2 = \partial U/\partial x, \quad \partial e U = U(x, t) - \text{force function} \tag{9.17}
\]

Write (9.17) in Hamiltonian form:

\[
dx/dt = \partial H/\partial y, \quad dy/dt = -\partial H/\partial x, \tag{9.18}
\]

where \( H = K - U \) – Hamilton’s function of the system,
\( K = (y, y)/2 \) – kinetic energy, \((\cdot, \cdot)\) – coordinate scalar product.

For the Hamiltonian system (9.18) the differential form \( \omega \), defining symplectic structure on the manifold \( M = \mathbb{R}^3 \times \mathbb{R}^3 \), has the form:

\[
\omega(\xi_1, \xi_2) = (x_1, y_1) - (y_2, x_2), \tag{9.19}
\]

where \( \xi_1 = (x, y; x_1', y_1') \), \( \xi_2 = (x, y; x_2', y_2') \in TM \)

Using the technique described above we can make transition to autonomous Hamiltonian system :

\[
(9.20)
\]

\[
H_a = H+T = (y, y)/2 - U + T,
\]

\[
dx/dt = \partial H_a/\partial y, \quad dy/dt = -\partial H_a/\partial x, \quad d\tau/dt = \partial H_a/\partial T, \quad dT/dt = -\partial H_a/\partial t.
\]

\[
\omega_a(\xi_1^a, \xi_2^a) = (x_1', y_1') - (y_2', x_2') + (\dot{t}_1', \dot{T}'_2) - (T'_1, \dot{t}_2'), \tag{9.20}
\]

where \( \xi_1^a = (x, t, y; x_1', y_1', T') \), \( \xi_2^a = (x, t, y, T; x_2', \dot{t}_2', y_2', T_2') \in TM_a \)

From the topological standpoint it will be more relevant to represent the phase space \( M \) for Earth satellite in the form: \( M = \mathbb{T}^3 \times [0, \pi] \times G \), where \( \mathbb{T}^3 \) – is three dimensional tore of the angles (\( M \) – mean anomaly, \( \omega \) - perigee argument, \( \Omega \) - longitude of ascending node), \([0, \pi]\) – is the parameter set for inclination \( i \), \( G = \{ (a,e) : a>R^e, 0 \leq e < R^e/a \} \), \( a \) – semi-major axis, \( e \) – eccentricity, \( R^e \) – Earth radius.

When we use the perturbation theory, the force function and the Hamilton’s function are written in the form:
\[ U = U_0 + \varepsilon U_1, \]
where \( U_0 = \mu E/r \) – the main part of geopotential, \( U_1 \) – function, accounting of the perturbing factors (earth oblateness, Lunar and Solar gravity etc.)
\[ \mu E \] – Earth gravitational constant,
\[ r \] – the length of satellite radius-vector.
\( H_a = H_0 + \varepsilon H_1, \quad (9.20) \)
where \( H_0 = (y, y) / 2 - U_0 + T = T - \mu E/(2a) \) – Hamilton’s function of non-perturbed (Keplerian) motion,
\[ H_1 = -U_1, \quad \varepsilon - small \ parameter \ of \ perturbation \ theory \ (\varepsilon = 1). \]

Other representations of the force function [9] are also possible.

9.2 Lidov’s method

Lidov’s method is used for solving the special type of differential equations in partial derivatives. Such equations arrive in applications of perturbation theory to the motion of satellites with account of Lunar and Solar gravity.

Consider the following form of the differential equation in partial derivatives:
\[ n \cdot \partial W/\partial M + \partial W/\partial t = F( M, t), \quad (9.21) \]
where \( F( M, t ) \) – \( 2\pi \)-periodic in \( M \) function with zero mean:
\[ \int_{2\pi} F( M, t ) \, dM = 0 \]

Thus follows that the indefinite integral \( G_t (M) = \int F( M, t ) \, dM \) is \( 2\pi \)-periodic function for any fixed \( t \).

Introduce linear operators \( \Phi \) and \( \Psi \):
\[ \Phi : T_0 [0, 2\pi] \rightarrow T[0, 2\pi] \quad \Phi(F) = \int_M F(\xi) \, d\xi / n; \quad (9.22) \]
\[ \Psi : T_0 [0, 2\pi] \rightarrow T_0 [0, 2\pi] \quad \Psi(F) = \Phi(F) - \int_{2\pi} \Phi( F ) (\xi) \, d\xi / (2\pi), \]

where \( T[0, 2\pi] \) – the set of integrable \( 2\pi \)-periodic functions,
\( T_0 [0, 2\pi] \) – functions from \( T[0, 2\pi] \) with zero mean.

Since operator \( \Psi \) commutes with the operator of partial differentiation\(^{13} \), the formal solution of (9.21) can be written in the form of a series:
\[ W = \sum_{m=0}^{\infty} (-1)^m \cdot \partial^m \Psi^{(m+1)}(F) / \partial^m t, \quad m = 0 \div \infty, \quad (9.23) \]
where \( \Psi^m(F) = \Psi(\Psi^{(m-1)}(F)). \)

Denote \( W_n \) the partial sum of the series (9.23) \( (m = 0 \div n) \). Due to equality:
\[ n \cdot \partial (\partial^m \Psi^{(m+1)}(F) / \partial^m t) / \partial M + \partial (\partial^m \Psi^{(m+1)}(F) / \partial^m t) / \partial t = \partial^m \Psi^m(F) / \partial^m t + \partial (\Psi^{(m+1)}(F) / \partial (m+1) t \]
we have \( n \cdot \partial W_n / \partial M + \partial W_n / \partial t = (-1)^n \cdot \partial^{(n+1)} \Psi^{(n+1)}(F) / \partial^{(n+1)} t \). If the right side of the last equality is small, then \( W_n \) provides good approximation for the solution of equation (9.21).

When we are going to determine the gravitational perturbations from the Moon and the Sun:

\[^{13} \text{This is valid for extensive class of functions, e.g. for functions smooth in } M \text{ and } t. \]
\[\partial^n\Psi^n(F)/\partial^n t \sim (nB/n)^n,\] where \(nB\) – the angular velocity of the motion of the perturbing body around the Earth. For satellites with periods \(\leq 1\) day and for the Moon - \(nB/n \sim 1/28\), for the Sun - \(nB/n \sim 1/365\). Hence the partial sums (9.23) can be used for representation of the solution of the equation (9.21).

If the right side of the equation (9.21) can be written in the form \(F(M, t) = A(M) \cdot B(t)\), the solution (9.23) can be written in the form:

\[W = \sum_m (-1)^m \cdot \Psi^{m+1}(A) \cdot \partial^m B/\partial^m t, \quad m = 0 \div \infty\]

Conclusions and acknowledgements.

We tried to consider all the principle ideas constituting the basis of the algorithm. The technical details of the implementation of these ideas were not discussed here.

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References

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Fig. 1.
Second Tesseral harmonic of geopotential.
Fig. 2
Determination of the perturbations from Tesseral harmonics.
The integrand function.