ORBIT DETERMINATION
OF LEO SATELLITES
FOR A SINGLE PASS
THROUGH A RADAR:
COMPARISON OF METHODS

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Introduction

Initial Orbit Determination (IOD)

• Russia radar sites track each object and report the state vector and covariance matrix as the observation instead of observations.

• Current IOD approaches were established in the 1960s and 70s. Very little interest until recently.

• The planned radar fence upgrade and WFOV cameras will grow the catalog from the current 15000 objects to an estimated 150000 objects. Correlating tracks from these objects will be a major challenge.

• The accuracy of collision probability determination depend on the accuracy state and covariances. Need improved state estimates and covariances from the IOD.
The least squares method search for the minimum of

\[ \Psi(a) = \sum_{k=1}^{n} \left( \frac{1}{\sigma_{d_k}^2} (d_k - d_k(a))^2 + \frac{1}{\sigma_{\alpha_k}^2} (\alpha_k - \alpha_k(a))^2 + \frac{1}{\sigma_{\beta_k}^2} (\beta_k - \beta_k(a))^2 \right), \]

where

- \( d_k, \alpha_k, \beta_k \) – k-th measurement range, azimuth, elevation \((k = 1, 2, \ldots, n)\);
- \( t_k \) – epoch of the k-th measurement;
- \( \sigma_{d_k}, \sigma_{\alpha_k}, \sigma_{\beta_k} \) – RMS of the errors \( d_k, \alpha_k, \beta_k \);
- \( a = a(\bar{t}) \) – state at epoch \( \bar{t} \).
- \( h_k(a) = (d_k(a), \alpha_k(a), \beta_k(a)) \) – values of the parameters of the k-th measurement, calculated using the vector \( a \).

The solution \( a_{\text{min}} \) is obtained from \( \left( \frac{\partial \Psi}{\partial a}(a_{\text{min}}) \right) = 0 \) and covariance \( K \) from \( K = 0.5 \cdot \left( \frac{\partial^2 \Psi}{\partial a^2}(a_{\text{min}}) \right)^{-1} \).
Recurrent algorithm

The estimate $a_{\text{min}}$ of the least squares method has recurrent structure: the estimate of parameters based on $k$ measurements can be written as a function of the same estimate for $k-1$ measurements and the $k$-th measurement.

The recurrence feature of the least squares estimate can be extended to more general case, when the parameters $a_k = a(t_k)$ of the system for the time $t_k$ (in our case the orbital parameters for the time $t_k$) in addition to deterministic component have a random one which has Markov’s feature (the ”future” for the fixed ”past” depends only on the ”present”), i.e.

$$ p(a_k|a_{k-1}, a_{k-2}, \ldots) = p(a_k|a_{k-1}) $$

(M)

where $p(a_k|\ldots)$ - conditional probability density $a_k$.

The condition (M) is satisfied if

$$ a_k = f_k(a_{k-1}) + \gamma_k $$

(M1)

where $\gamma_k$ - ”noise of the system” - time independent random perturbations with zero mean and covariation matrices $\Gamma_k$. 
For (M), (M1) and independent errors of the measurements for the a posteriori (after acquisition of the measurement $x_k$) probability density of the parameters $a_k$ we have recurrent formulas for density $p$ the first and second moments $a, P$

$$p(a_k|x_k, a_{k-1}, a_{k-2}, ...) = p(a_k|x_k, a_{k-1}) = p(x_k|a_k) p(a_k, a_{k-1})/p(x_k)$$

$$P_k^{-1} = P_{k|k-1}^{-1} + H_k' R_k^{-1} H_k$$  
$$P_k^{-1} \hat{a}_k = P_{k|k-1}^{-1} \hat{a}_{k|k-1} + H_k' R_k^{-1} x_k,$$

$$\hat{a}_{k|k-1} = f_k(\hat{a}_{k-1})$$  
$$F_k = \frac{\partial f_k(\hat{a}_{k-1})}{\partial \hat{a}_{k-1}}$$  
$$H_k = \frac{\partial h_k(\hat{a}_{k|k-1})}{\partial a_k}$$

$$P_{k|k-1} = F_k P_{k-1} F_k' + \Gamma_k$$

If measured parameter is a scalar $u$

$$\tilde{w}_k = w_k/(1 + w_k h_k' P_{k|k-1} h_k)$$

$$P_k = P_{k|k-1} - \tilde{w}_k (P_{k|k-1} h_k) (P_{k|k-1} h_k)'$$

$$\hat{a}_k = \hat{a}_{k|k-1} + w_k P_k h_k (u_k - h_k(\hat{a}_{k|k-1}))$$

where $w_k = 1/\sigma_{u_k}^2$ — is the weight characteristic of measurement $u_k$. 
Least Squares and Recurrent Issues

- Selection of the state \( a \), the dynamic model and the propagation method for state and covariances.

- Calculation of the initial approximaiton \( \hat{a}_0, P_0 \) for the recurrent algorithm,

- The technique for calculating the noise matrix of the system \( \Gamma_k \) for the recurrent algorithm.

- Calculation of the initial approximaiton \( a_0 \), for minimization of \( \Psi(a) \);

- Selection of technique for reaching the minimum of \( \Psi(a) \).
The state vector $\vec{a}$

$\vec{a} = (x, y, z, \dot{x}, \dot{y}, \dot{z})$ for the certain time in the local rectangular coordinate frame (LRCF) related to geographical directions. The basic plane is the plane of local horizon, the $x$ axis is in the plane of local horizon and is directed to the west, the $y$ axis is directed normal to the plane of local horizon upwards ad $z$ axis is in the plane of local horizon and is directed to the north.

\[
x = -d \sin \alpha \cos \beta \\
y = d \sin \beta \\
z = d \cos \alpha \cos \beta
\]
The dynamic model $\vec{d}(t)$

$$\ddot{\vec{d}} = -\frac{\mu}{r^3}\vec{r} - 2\vec{\omega}\times\dot{\vec{d}} - \vec{\omega}\times(\vec{\omega}\times\vec{r}) - \frac{3J_2\mu R_e^2}{2r^5}\left(\vec{r} + 2(\vec{r},\vec{\omega}_0)\vec{\omega}_0 - \frac{5(\vec{r},\vec{\omega}_0)^2}{r^2}\vec{r}\right)$$

$\vec{d} = (x, y, z)'$, $\dot{\vec{d}} = (\dot{x}, \dot{y}, \dot{z})'$

$r = (x-b_x, y-b_y, z-b_z)'$ – Greenwich coordinates of SO

$$r = \sqrt{(x-b_x)^2 + (y-b_y)^2 + (z-b_z)^2}$$

$$r_1 = \sqrt{X_{g0}^2 + Y_{g0}^2} \quad r_0 = \sqrt{X_{g0}^2 + Y_{g0}^2 + Z_{g0}^2}$$

$$\ddot{z} = Z_{g0}/r_0 \quad \ddot{r}_1 = r_1/r_0 \quad R = R_e \cdot (1 - \alpha \cdot \ddot{z}^2)$$

$$b_x = 0 \quad b_y = -r_0 \quad b_z = 2\cdot\alpha\cdot R \cdot \ddot{z} \cdot \ddot{r}_1$$

$$\vec{\omega} = \omega_e \cdot (\omega_{0x}, \omega_{0y}, \omega_{0z}) \quad \omega_e = 0.00072921158 \quad 1/s$$

$$\omega_{0x} = 0 \quad \omega_{0y} = \ddot{z} + \ddot{r}_1 \cdot b_z/r_0 \quad \omega_{0z} = \ddot{r}_1 - \ddot{z} \cdot b_z/r_0$$

$$\alpha = 0.003353 \quad \mu = 398600.44 \ km^3/s^2 \quad J_2 = 0.001083 \quad R_e = 6378.137 \ km$$

$X_{g0}, Y_{g0}, Z_{g0}$ – Greenwich coordinates of LRCF origin
The propagation method for state and covariances.

For solving the system of differential equations $\ddot{\vec{d}} = f(\vec{d}, \dot{\vec{d}})$ use the numerical Runge-Cutta method of the 4th order.

$$\Delta a_i = \frac{(k_1^{(i)} + 2k_2^{(i)} + 2k_3^{(i)} + k_4^{(i)})}{6}$$

$$k_1^{(i)} = \tau_i \cdot f(a_i) \quad k_2^{(i)} = \tau_i \cdot f(a_i + 0.5k_1^{(i)})$$

$$k_3^{(i)} = \tau_i \cdot f(a_i + 0.5k_2^{(i)}) \quad k_4^{(i)} = \tau_i \cdot f(a_i + k_3^{(i)})$$

For the operator $F_{k}(a_{k-1})$ we will take the matrix $F_{k}$ with dimensions $6 \times 6$ with the following non-zero elements: $f_{i,i} = 1$ for $i = 1, 2, ..., 6$, $f_{i,i+3} = \tau_k$ for $i = 1, 2, 3$, where $\tau_k = t_k - t_{k-1}$.

$$F_n = \begin{pmatrix}
1 & 0 & 0 & \tau_n & 0 & 0 \\
0 & 1 & 0 & 0 & \tau_n & 0 \\
0 & 0 & 1 & 0 & 0 & \tau_n \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}$$
Calculation of the initial approximation $\hat{a}_0$, $P_0$ for the recurrent algorithm,

The state and covariance estimate is determined from the 1st two observations $d_1$ and $d_2$.

The technique for calculating the noise matrix of the system $\Gamma_k$ for the recurrent algorithm.

$$\Gamma_{11,k} = \begin{pmatrix}
110\cdot \sigma_d^2/(\tau_n^2(\Delta t_{ef,d})^4) & 0 & 0 \\
0 & 110\cdot \sigma_\alpha^2/(\tau_n^2(\Delta t_{ef,\alpha})^4) & 0 \\
0 & 0 & 110\cdot \sigma_\beta^2/(\tau_n^2(\Delta t_{ef,\beta})^4)
\end{pmatrix}$$

$$\Gamma_k = \begin{pmatrix}
0 & 0 \\
0 & \Gamma_{11,k}
\end{pmatrix}$$

where

- $\sigma_d, \sigma_\alpha, \sigma_\beta$ – are the RMS values of typical values of the errors of measured parameters,
- $\Delta t_{ef,d}, \Delta t_{ef,\alpha}, \Delta t_{ef,\beta}$ – efficient memory of the algorithm for parameters $d, \alpha, \beta$ (parameters of the algorithm selected on experimental basis).
Calculation of the initial approximation $a_0$, for minimization of $\Psi(a)$;

For the initial approximation $a_0$ we use the estimate $\hat{a}_n$, obtained using measurements $x_1, x_2, ..., x_n$ by the recurrent algorithm.

Selection of technique for reaching the minimum of $\Psi(a)$.

The major effect of updating by least squares is the better accuracy of the velocity components $\dot{u}=(\dot{x}, \dot{y}, \dot{z})$ of the state vector $a$. The position parameters $u=(x, y, z)$ of the vector $a$ in fact are not updated. Thus we suggest to make minimization of $\Psi(a) = \Psi(u, \dot{u}) = \psi(\dot{u})$ only by the vector $\dot{u}$.

For the point of the minimum $\dot{u}_{\text{min}}$ of the function $\psi(\dot{u})$ for the case when it is reached by one iteration we suggest the Newton’s formula

$$\dot{u}_{\text{min}} = \dot{u}_0 - \left( \frac{\partial^2 \psi}{\partial \dot{u}^2}(a_0) \right)^{-1} \frac{\partial \psi}{\partial \dot{u}}(a_0)$$

where the first and the second derivatives of the function $\psi(\dot{u})$ with respect to parameters $\dot{x}, \dot{y}, \dot{z}$ are calculated by finite differences method.
New (Guarantee) Approach

- Fundamental problem with the traditional methods (joint and recurrent) is the assumption of uncorrelated measurements and known measurement statistics.

- Because there is no a priori orbit identifying "bad" measurements can be difficult.

- Basis of the "Guarantee" method is a guaranteed range of the measurement errors. "Guarantee" means that the algorithm produces not only the orbit estimate but the maximum possible errors in the estimated parameters.

- Leads to a more accurate estimate of the orbit and uncertainty. Provides an upper bound on the state errors.

- More computationally intensive. This feature hindered early use.
Let for the moments \( t_k \) \( (t_k \leq t_{k+1}; \ k = 1, 2, \ldots n) \) we acquire measurements \( u_k \) of certain functions \( h_k(c) \) of the m-vector of parameters \( c \) \( (m<n) \), and the errors of the measurements \( \delta u_k = u_k - h_k(c) \) are limited from above by constant \( \delta_{k,\text{max}} \).

The limits for errors of the measurements define in the m-dimensional space of parameters \( c = (c_1, c_2, \ldots, c_m) \) a domain of possible values of \( c \)

\[
D_n = \bigcap_{k=1}^{n} \{ u_k - \delta i,\text{max} \leq h_k(c) \leq u_k - \delta k,\text{max} \}
\]

Project this domain to the coordinate of the components of vector \( c \) and among the projected points for each axis find the most right \( c_{n,r} \) and the most left \( c_{n,l} \).

They define the boundaries (maximum and minimum values) for the changes of each component of parameter \( c \). In this case the estimate \( \bar{c}_n \) of parameter and maximum errors \( \delta \bar{c}_{n,\text{max}} \) of this estimate are naturally defined as

\[
\bar{c}_n = \frac{1}{2} (c_{n,r} + c_{n,l}) \quad \delta \bar{c}_{n,\text{max}} = \frac{1}{2} (c_{n,r} - c_{n,l})
\]

When the measured parameters are linearly related to the determined parameter \( a \), finding \( \bar{c}_n \) and \( \delta \bar{c}_{n,\text{max}} \) is a linear programming problem.
The Model Example

\[ m = 1 \quad h(c) = c \quad u_k = c + \delta_k, \quad \delta_{k,max} = \delta_{max}. \]

\[ D_n = \bigcap_{k=1}^{n}\{[u_k-\delta_{max}, u_k+\delta_{max}]\} = [\max_k u_k-\delta_{max}, \min_k u_k+\delta_{max}] \]

\[ \bar{c}_n = \frac{1}{2}(\max_k u_k + \min_k u_k) \quad \delta_{\bar{c}_{n,max}} = \delta_{max} - \frac{1}{2}(\max_k u_k - \min_k u_k) \]

\[ [a, b] \quad \text{interval with left end } a \text{ and right end } b. \]

• \( \bar{c}_n \) does not depend on \( \delta_{max} \)

• If we know \( \delta_{max} \) precisely the value \( \delta_{\bar{c}_{n,max}} \) is a correct upper estimate for the error of the estimate \( \bar{c}_n \), for any correlations of the errors of different measurements.

• For not correlated errors of measurements the estimate \( \bar{c}_n \) with regard to accuracy is not inferior and sometimes can be essentially more accurate (!) then the estimate \( \hat{c}_n = \frac{1}{n} \sum_{k=1}^{n} u_k \) of the least squares technique.

  – for uniform distribution:

  \[ \sigma_{\bar{c}_n} \approx \frac{1.4 \cdot \delta_{max}}{\sqrt{n}} \quad \sigma_{\hat{c}_n} \approx \frac{0.58 \cdot \delta_{max}}{\sqrt{n}} \quad (1) \]

  – for triangular distribution:

  \[ \sigma_{\bar{c}_n} \approx \frac{0.46 \cdot \delta_{max}}{\sqrt{n}} \quad \sigma_{\hat{c}_n} \approx \frac{0.41 \cdot \delta_{max}}{\sqrt{n}} \quad (2) \]
The Real Algorithm

Assume there are measurements range \( d_k \), azimuth \( \alpha_k \) and elevation \( \beta_k \) at the times \( t_k \) \((k = 1, 2, ..., n)\). The error limits are defined be \( \delta_{d,\text{max}} \), \( \delta_{\alpha,\text{max}} \), \( \delta_{\beta,\text{max}} \). Let epoch be \( \bar{t} = 0.5 \cdot (t_1 + t_n) \) and orbital parameters are \( c = (d, \alpha, \beta, \dot{d}, \dot{\alpha}, \dot{\beta}) \).

Divide the measurements into \( n/2 \) groups. Each group defines two points. The k-th group is \((t_k, t_{0.5n+k})\).

From two points at two different times determine the state or orbit defined by \( \hat{c}_k = (\hat{d}, \hat{\alpha}, \hat{\beta}, \hat{\dot{d}}, \hat{\dot{\alpha}}, \hat{\dot{\beta}})_k \). The determination of the states \( \hat{c}(\bar{t}) \) assume 2-body motion. Now compute the measurements that would give this solution including \( J_2 \). Integrate the equation for the deviations of \( \Delta r, \Delta \dot{r} \) from Keplerian orbit with the initial conditions \( \Delta r(\bar{t}) = \Delta \dot{r}(\bar{t}) = 0 \) and \( r = r_k \) to obtain the errors in the measurements \( \Delta r_k(t_k) \). Then compute the corrected measurements \( \tilde{r}_k = r_k - \Delta r_k \) and transform the \( \tilde{r}_k \) to the measurement frame.

The domain \( D_k \) is approximated by a six-dimensional parallelepiped with the center at \( \hat{c}_k \) and 64 apexes determined by

\[
\hat{c}_k \pm \delta_{d,\text{max}} \cdot j_1 \pm \delta_{\alpha,\text{max}} \cdot j_2 \pm \delta_{\beta,\text{max}} \cdot j_3 \pm \delta_{d,\text{max}} \cdot j_4 \pm \delta_{\alpha,\text{max}} \cdot j_5 \pm \delta_{\beta,\text{max}} \cdot j_6
\]

where \( j_i \) \((i = 1, 2, ..., 6)\) are the lines of the matrix of partial derivatives \( J = \frac{\partial(c_k)}{\partial(u_k, u_{0.5n+k})} \). The boundary projections \( c_{k,l} \) and \( c_{k,r} \) of these 64 points determine the boundaries of the vector interval \( (c_{k,l}, c_{k,r}) \). Then

\[
(c_l, c_r) = \bigcap_{k=1}^{n/2} (c_{k,l}, c_{k,r})
\]

\[
\bar{c}_n = \frac{1}{2}(c_{n,r} + c_{n,l})
\]

\[
\delta \bar{c}_{n,\text{max}} = \frac{1}{2}(c_{n,r} - c_{n,l})
\]
References


• Obtaining the best initial orbit is vital for correlating tracks and developing the future space catalog.

• Philosophy and approach US and Russian methods of initial orbit determination are different. Current Russian methods of initial orbit determination have been reviewed.

• New method called The ”Guarantee” method has been developed.
  • Provides a more accurate orbit estimate.
  • Provides upper bound on estimated state error.
  • More computationally intensive.