Accuracy of orbit determination and prediction for SOs in LEO.
Dependence of estimate errors from accuracy and number of measurements
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Abstract. The investigation of the accuracy for along track parameters of orbit determination for LEO space objects (SOs) is performed on a developed model. The results testify to a very strong effect of system’s noise level and processing interval on the accuracy of estimates of a state vector. For a wide scope of SOs, which are subject to the atmospheric drag, the achievable RMS deviations of errors in the determination of drag characteristics lie in the range from 2% to 12%, depending on the variability of drag and “signal-to-noise ratio”. These estimates rather well correlate with the experience of the determination of drag characteristics for various SOs. The estimates of orbit prediction errors as a function of the basic influencing factors are presented.

1. Comparison of various approaches to SO state vector estimation

Now we consider the problem of estimating the SO state vector $x \ (n \times 1)$ from measurements $Z \ (k \times 1)$ in the classical formulation. We shall take into account the possibility of the existence of some nuisance (disturbing) parameters $q \ (m \times 1)$. In this case the basic initial relation is as follows:

$$Z = X \cdot x + B \cdot q + V.$$  \hspace{1cm} (1)

Here $X \ (k \times n)$ and $B \ (k \times m)$ are known matrices, $V \ (k \times 1)$ is the vector of measurement errors, which are accepted to be of equal accuracy and statistically independent, i. e.

$$M \{V \cdot V^T\} = \sigma_z^2 \cdot E.$$  \hspace{1cm} (2)

The correlation matrix $M \{q \cdot q^T\} = \sigma_q^2 \cdot K_q$ of nuisance parameters is supposed to be known. We shall consider three approaches to state vector estimation, which differ in the technique of accounting for nuisance parameters:

I. Without accounting for nuisance (disturbing) parameters. In the process of state vector estimation, the influence of these parameters is not taken into account. In this case the classical least-square technique (LST) is applied for estimation:

$$\hat{x} = \left(X^T \cdot X\right)^{-1} \cdot X^T \cdot Z.$$  \hspace{1cm} (3)

It can easily be shown that the correlation matrix of state vector errors $K_x$ is expressed as follows:

$$K_x = \sigma_z^2 \cdot \left(X^T \cdot X\right)^{-1} + \left(X^T \cdot X\right)^{-1} \cdot X^T \cdot \left(B \cdot K_q \cdot B^T\right) \cdot X \cdot \left(X^T \cdot X\right)^{-1}.$$  \hspace{1cm} (4)

II. Parameterization. The state vector of nuisance (disturbing) parameters is introduced in the structure of an extended state vector $y^T = [x \ q]^T$, and then the LST is applied. In this case the required estimate and its correlation matrix are expressed as follows:

$$\hat{y} = \left[\hat{x} \ \hat{q}\right]^T = \left[\left(X^T \cdot X\right)^{-1} \cdot X^T \cdot B \cdot K_q \cdot B^T \cdot X \cdot \left(X^T \cdot X\right)^{-1}\right] \cdot Z,$$  \hspace{1cm} (5)

$$K_y = \begin{bmatrix} K_{11} & K_{12} \\ K_{12}^T & K_{22} \end{bmatrix} = \sigma_z^2 \cdot \left[\left(X^T \cdot X\right)^{-1} \cdot X^T \cdot B \cdot K_q \cdot B^T \cdot X \cdot \left(X^T \cdot X\right)^{-1}\right]^{-1}.$$  \hspace{1cm} (6)

III. Without parameterization (the optimum filtering of measurements). The a priori correlation matrix of nuisance (disturbing) parameters is used for “weighing” the measurements without extension of a state vector. The influence of these parameters is taken into account by combining
them with measurement errors \( V = B \cdot q + V \), and then the maximum-likelihood method (MLM) is applied. In this case the required estimate and its correlation matrix are expressed as follows:

\[
\hat{X} = \left( X^T \cdot P \cdot X \right)^{-1} \cdot X^T \cdot P \cdot Z,
\]

\[
P = \left( \frac{\sigma_q^2}{\sigma_z^2} \cdot B \cdot K \cdot B^T + E \right)^{-1} = \left( S_n^2 \cdot B \cdot K \cdot B^T + E \right)^{-1}.
\]

\[
K_x = \sigma_z^2 \cdot \left( X^T \cdot P \cdot X \right)^{-1}.
\]

Here parameter \( S_n \) can be treated as the signal-to-noise merit.

As a result of the performed analysis, the comparative relationships were established between state vector errors using the techniques listed above. The results of the analysis are presented in the figure.

![Figure 1. Dependence of errors on the applied approach](image)

It is seen that there exists a level of (small) disturbances, for which it is more profitable to apply the LST without extension of a state vector. However, even in this case the errors are greater, than in case of using the non-parametric approach.

The results of the investigation of the first and third approaches on the model are considered in detail. In this case the statistical characteristics of the atmospheric disturbances are used.

2. Technique for estimating the forecasting errors

In a number of papers (Refs. 1-4) the authors have successfully applied simplified equations of motion for evaluating the influence of disturbances on time parameters (the errors along the trajectory). In this case the state vector includes only the orbital elements, which characterize the motion in the plane of a near-circular orbit. Following this approach, we shall consider the angular motion of a satellite in the orbital plane with the revolution number as an argument. The state vector components are the following three parameters: the equator crossing time \( i_1 \), the satellite revolution period \( T \), and the period change under an effect of disturbances per one revolution \( \Delta T \). The evolution of these parameters is described by the following equations:

\[
t_{i+1} = t_i + T,
\]

\[
T_{i+1} = T_i + \Delta T_i,
\]

\[
\Delta T = \Delta T_m \cdot (1 + q_i).
\]

Quantity \( \Delta T_i \) is supposed to be associated with the atmospheric drag effect only. In this case, with constant ballistic coefficient of a spacecraft, the value of \( \Delta T_i \) is proportional to the current
atmospheric density. Quantity $\Delta T_m$ is the mean value of $\Delta T_i$, and $q_i$ is the Gaussian random process with known correlation function $K_q(t, \tau)_0$.

The values of equator crossing time at the beginning of revolution $N_i$:

$$z_i = t_i + \nu_i, \ (i=1,2,...),$$

(11)

are used as measurements. Here the measurement errors $\nu_i$ (of discrete white noise type) are distributed according to the normal law with zero mathematic expectancy and variance $\sigma^2_z$. The time interval between the measurements is assumed to be constant and equal to $\Delta N$ (in revolutions).

The above equations (10), measurements (11) and statistical characteristics of random processes $q_i$ and $\nu_i$ allow one to determine the statistical characteristics of errors of determining and forecasting the time parameters. We shall consider, as the methods of determination of orbital parameters, the classical least-square technique (LST) and its generalization (the optimum filtering of measurements) that takes into account the effect of disturbances as a color noise (refs. 4, 5).

The algorithms and results of investigations, based on the aforementioned approach, are presented below. The orbit determination and forecasting technique, based on joint processing of measurements, is accepted as the basic one. In the process of investigations we have used the following auto correlation function of random process $q_i$:

$$K_q(t, \tau)_0 = \begin{cases} \sigma^2_q \left(1 - \frac{|t - \tau|}{\Delta}\right), & \text{by } |t - \tau| < \Delta, \\ 0 & \text{by } |t - \tau| \geq \Delta. \end{cases}$$

(12)

Using the least-square technique the problem is solved by statistical modeling. As an example, figure 2 presents the random discrete sequence of values $\Delta T_i$ for $\sigma_q \cdot \Delta T_m = 0.005$ min, $\Delta T_m = 0.015$ min, and $\Delta = 50$. These estimates are connected by solid lines.

![Figure 2. Random sequence of $\Delta T_i$ values](image-url)
Then, on the basis of results of modeling the random sequence \( q_i \), of some initial conditions \( t_0, T_0, \Delta T_m \) and equations (10), the sequence of state vector values \( x_i^T = [i, T_i, \Delta T_i] \) was calculated (with a step of one revolution).

The next operation of modeling the time parameters of an orbit is the calculation of a sequence of model values of measurements \( z_j \) by formula (11). In this case the random errors of measurements are determined by means of the random-number generator, and the constant time interval between the measurements \( \Delta N \) (in revolutions) is taken into account.

Thus, the results of modeling the time parameters of an orbit and the results of corresponding measurements depend on the values of the following initial parameters \( t_0, T_0, \Delta T_m, \sigma_q, \Delta, \sigma_z, \Delta N \) (seven parameters in total). Figure 3 presents the scheme of successive application of LST at processing measurements (11) during modeling.

Figure 3 Scheme of successive processing of measurements

This figure presents two time intervals of determination and forecasting of orbital parameters: the current interval (for the \( jd \)-th updating) and the subsequent interval constructed by shifting all data to \( dN \) revolutions. The following designations are applied here: \( nz \) – the number of measurements used at updating, \( np \) – the number of forecasts, \( dNp \) – the time interval (in revolutions) between successive forecasts. The black font indicates the numbers of measurements, and the red one – the numbers of forecasts. The maximum forecasting interval equals \( dNp \cdot np \) revolutions. The blue font indicates serial numbers of revolutions. The performance of corrections and forecasting according to the given scheme of their cyclic organization allows one to obtain a rather great number of realizations for acceptable time.

The classical formula of the least-square technique (3) is used to calculate the state vector estimates based on the measurements.

\[
Z_k = \begin{bmatrix}
  z_1 \\
  z_2 \\
  \vdots \\
  z_k
\end{bmatrix} - \text{is the vector of measurements.}
\]

The block matrix \( X = X_j \) is constructed on the basis of an analytical solution of the system of equations (10),

which for arbitrary time instant \( t_i \) (here \( i \) is the revolution number) and with a vector of initial conditions at arbitrary point \( t_j \) has the following form:

\[
t_i = t_j + (i - j) \cdot T_j + \frac{1}{2} \cdot (i - j) \cdot (i - j + 1) \cdot \Delta T_m .
\]

Therefore, the \( i \)-th line of the block matrix \( X_j \) is expressed as follows:
The number of lines in matrix $X_j$ is equal to the number of measurements $n_z$. The choice of the second argument ($t_j$) in this formula is of no principal significance. It does not influence the state vector estimates at forecasting points. In the algorithm, the instant $t_1$, corresponding to the first measurement $z_1$ in the current file of measurements, is chosen as this initial time instant.

The results of updating the state vector (3) from measurements appear to be dependent not only on seven parameters listed above, but also on one more additional parameter – the number of measurements $n_z$. At evaluating the errors of state vector estimations over the updating interval the results of estimations appear to be independent of the initial values of time ($t_0$) and the period ($T_0$). At evaluating the forecasting errors one more influencing factor – the forecasting interval $dNp\cdot np$ – is added. So, the total number of factors, influencing the value of errors, is equal to seven. They represent the following parameters: $\Delta T_m$, $\sigma_q$, $\Delta$, $\sigma_z$, $AN$, $n_z$ and $Np\cdot np$.

Below, in the process of studying the errors, the number of influencing factors, which are taken into account, is reduced, because some of them are attributed by fixed values corresponding to their observed real values.

As a result of these assumptions, three factors, among the basic influencing factors, are taken into account in the analysis of errors:
- Standard deviation of color noise $\sigma_q \cdot \Delta T_m$.
- Standard deviation of measurement errors $\sigma_z$.
- Number of processed measurements $n_z$.

In the analysis of the errors of the state vector estimations, $\delta \vec{x} = \vec{x} - \vec{x}$, it is convenient to use the normalized errors. This allows one to reduce the number of basic influencing factors down to two. Instead of parameters $\sigma_q \cdot \Delta T_m$ and $\sigma_z$, their ratio is used, i.e.

$$S_n = \frac{\sigma_q \cdot \Delta T_m}{\sigma_z},$$

which has a meaning of the “Signal-to-noise merit” characteristic.

In using the method of optimum filtering of measurements (OFM), which takes into account statistical characteristics of disturbances (Refs. 5, 6, 7), the RMS errors of orbital parameters were determined by the formula (9) for the correlation matrix of state vector errors. This expression has a traditional form and is applicable for any instant. The distinction consists in forming the weight matrix $P_j$ that takes into account not only the effect of measurement errors, but also the effect of system’s noises. The weight matrix $P_j$ is calculated by formula

$$P_j = \left[ \frac{\sigma^2_z}{S_n^2 \cdot Q_j + E} \right]^{-1}.$$  

This formula represents concrete definition of formula (5). Matrix $Q_j$ is constructed with allowance for the correlation function $K_q(t, \tau)_0$ and solution (13) of equation (10). In the absence of noise ($S_n = 0$) expressions for $P_j^{-1}$ and (2) coincide.

### 3. Influence of fit span upon the estimation accuracy

In using the LST the interval of processing (the fit span) is chosen depending on the type of satellite (the level of noises), as well as on the accuracy and quantity of measurements. We shall consider the influence of the fit span for the example of modeling the process of processing and forecasting the spacecraft motion with the following parameters:
RMS of time measurement errors $\sigma_z = 0.0002 \text{ min} = 0.012 \text{ sec};$
Time interval between the measurements (in revolutions) $dN=2;$
Mean value of period change per revolution $\Delta T_m = 0.002 \text{ min};$
Interval of atmospheric noise correlation $\Delta = 50 \text{ revolutions};$
RMS of normalized drag variations $\sigma_q = 0.1.$

The "signal-to-noise merit" $S_n = \sigma_q \cdot \Delta T_m / \sigma_z = 1.0$ corresponds to these parameters. The given version is considered below as the basic one. In studying the influence of various factors on the estimation accuracy the basic parameters ($\sigma_z, dN, \Delta T_m$) are varied separately.

Modeling was carried out for the number of measurements $nz = 4, 5, 6, \ldots, 11.$ Here the lower boundary should be greater, than the number of updated parameters (3 parameters). For each of the versions the number of realizations at statistical modeling was accepted to be 2000.

We have used, as optimization criteria, the data on RMS of time errors of determining the orbital parameters at the last measurement instant (Sig (0)), as well as at forecasting for 15 (Sig (15)) and 30 (Sig (30)) revolutions. The results of calculations are presented in table 1. RMS errors are presented both in the non-normalized and normalized form.

Table 1. Influence of a fit span upon the accuracy of orbital parameters

<table>
<thead>
<tr>
<th>nz</th>
<th>Sig(0)</th>
<th>Sig(15)</th>
<th>Sig(60)</th>
<th>Sig(0)</th>
<th>Sig(15)</th>
<th>Sig(60)</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>0.012</td>
<td>0.793</td>
<td>15.945</td>
<td>1.019</td>
<td>0.588</td>
<td>0.738</td>
</tr>
<tr>
<td>5</td>
<td>0.011</td>
<td>0.761</td>
<td>16.325</td>
<td>0.937</td>
<td>0.564</td>
<td>0.756</td>
</tr>
<tr>
<td>6</td>
<td>0.011</td>
<td>0.775</td>
<td>15.755</td>
<td>0.949</td>
<td>0.574</td>
<td>0.729</td>
</tr>
<tr>
<td>7</td>
<td>0.013</td>
<td>0.981</td>
<td>21.242</td>
<td>1.078</td>
<td>0.727</td>
<td>0.983</td>
</tr>
<tr>
<td>8</td>
<td>0.016</td>
<td>1.027</td>
<td>20.036</td>
<td>1.325</td>
<td>0.761</td>
<td>0.928</td>
</tr>
<tr>
<td>9</td>
<td>0.026</td>
<td>1.128</td>
<td>20.280</td>
<td>2.140</td>
<td>0.836</td>
<td>0.939</td>
</tr>
<tr>
<td>10</td>
<td>0.039</td>
<td>1.483</td>
<td>23.443</td>
<td>3.215</td>
<td>1.098</td>
<td>1.085</td>
</tr>
</tbody>
</table>

It is seen from the presented data, that there exists some optimum measurement interval ($nz=5$), for which the minimum values of RMS of errors of orbital parameters determination are achieved. Small deviations from the general regularity are observed in the accuracy characteristics, which, in the majority of cases, are at the level of several percents and are explained by a limited number of realizations at modeling.

For non-optimal values of measurement interval the greatest distinctions in RMS errors are observed for accuracy characteristics at the last measurement instant, and the lowest ones – for characteristics of forecasting for 60 revolutions. In the given case the RMS of lowest errors at the last measurement instant are achieved with a small number of measurements. These estimates are close to the RMS of measurement errors ($\sigma_z = 0.012 \text{ sec}).$

4. Dependence of the optimum fit span from the noise level

The level of atmospheric noises depends, basically, on the value of characteristics of the satellites’ drag in the atmosphere. We shall choose the mean value of period change per revolution ($\Delta T_m$) on the basis of the data on various satellites’ drag at various altitudes (Figure 4).

The process of spacecraft motion processing and forecasting was modeled for the following values of parameters:
- Time interval between the measurements (in revolutions) $dN=2;$
- Interval of atmospheric noise correlation $\Delta = 50 \text{ revolutions};$
- RMS of normalized drag variations $\sigma_q = 0.1.$
Two versions of RMS errors of time measurements were considered: $\sigma_z = 0.0002 \text{ min} = 0.012 \text{ sec}$ and $\sigma_z = 0.001 \text{ min} = 0.06 \text{ sec}$. The possible range of altitudes of satellites affected by drag in the atmosphere was accepted to be 300 – 800 km.

The possible range of altitudes of satellites affected by drag in the atmosphere was accepted to be 300 – 800 km.

The results of the determination of the optimum measurement interval are presented in Table 2. The same table also presents three characteristics of space object’s (SO) drag at various altitudes: the values of period change per revolution (min/revolution and min/day), as well as the corresponding value of quantity $\sigma q \cdot \Delta T_m$.

Table 2. Fit span (number of measurements / day) for SOs at various altitudes and for various accuracy of measurements

<table>
<thead>
<tr>
<th>Altitude, km</th>
<th>300</th>
<th>400</th>
<th>500</th>
<th>600</th>
<th>700</th>
<th>800</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta T_m$, min/rev</td>
<td>2.0E-3</td>
<td>0.6E-3</td>
<td>2.5E-4</td>
<td>1.0E-4</td>
<td>4.0E-5</td>
<td>1.5E-5</td>
</tr>
<tr>
<td>$\Delta T_m$, min/day</td>
<td>0.030</td>
<td>0.009</td>
<td>0.0037</td>
<td>0.0015</td>
<td>0.0006</td>
<td>0.00022</td>
</tr>
<tr>
<td>$\sigma q \cdot \Delta T_m$, min</td>
<td>2.0E-4</td>
<td>0.6E-4</td>
<td>2.5E-5</td>
<td>1.0E-5</td>
<td>4.0E-6</td>
<td>1.5E-6</td>
</tr>
</tbody>
</table>

It is seen from these data, that under considered conditions (dN=2, two versions of RMS errors of measurements) the fit span increases about 5 times, as SO altitudes increase from 300 to 800 km. With 5-fold worsening of measurement accuracy, the fit span increases about 1.5-2 times. For the considered versions of initial data the optimum values of a fit span lie in the range from 0.6 to 6 days.

5. Dependence of RMS forecasting errors from the noise level (altitude)

We shall consider the results of estimations of RMS of time errors of forecasting for the following values of parameters:

- Time interval between the measurements (in revolutions) dN=2;
- RMS of time measurement errors $\sigma_z = 0.0002 \text{ min} = 0.012 \text{ sec}$;
- Interval of atmospheric noise correlation $\Delta = 50$ revolutions;
RMS of normalized drag variations $\sigma_q = 0.1$.

The possible range of altitudes of satellites affected by drag in the atmosphere was accepted to be 300 – 800 km (as in Section 3). For these initial data versions the greatest value of the "signal-to-noise merit" $S_M^q = \sigma_q \cdot \Delta T_m / \sigma_z = 1.0$ relates to the altitude of 300 km. As the altitude increases in the considered range, this ratio decreases by a factor of 130. The estimates of RMS of errors at the initial point (for the last measurement instant), as well as at forecasting for 15 and 60 revolutions, are presented in Table 3. They relate to application of both LST (for the optimum measurement interval) and the technique of Optimal Filtration of Measurements (OFM), which takes into account the effect of noises in calculating the weight matrix. The same results are presented in Figures 5, 6 and 7.

Table 3. RMS of forecasting errors (sec) for SOs at various altitudes, dN=2

<table>
<thead>
<tr>
<th>Altitude, km</th>
<th>LST by nz=opt</th>
<th>OFM, nz=(15) - 30</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Sig(0)</td>
<td>Sig(15)</td>
</tr>
<tr>
<td>300</td>
<td>0.011</td>
<td>0.761</td>
</tr>
<tr>
<td>400</td>
<td>0.011</td>
<td>0.320</td>
</tr>
<tr>
<td>500</td>
<td>0.0099</td>
<td>0.172</td>
</tr>
<tr>
<td>600</td>
<td>0.0092</td>
<td>0.090</td>
</tr>
<tr>
<td>700</td>
<td>0.0078</td>
<td>0.046</td>
</tr>
<tr>
<td>800</td>
<td>0.0069</td>
<td>0.027</td>
</tr>
</tbody>
</table>

The decrease of atmospheric noise (with increasing SO’s altitude) had the greatest effect on the errors of forecasting for 60 revolutions. They decreased by a factor of 75. In this case the RMS errors at the last measurement instant decreased 1.5 times only. It is also seen from the data presented above, that under considered conditions the application of LST (for the optimum measurement interval) and OFM results in virtually identical accuracy of estimates. It was shown above, that the deviation from the optimum measurement interval results, with using LST, in worsening the accuracy of estimations. Such an effect is absent in the case of using the OFM technique.

Figure 5. RMS of errors at the last measurement instant, sec
Table 4. Relative errors of mean drag values at various altitudes and for various accuracy of measurements

<table>
<thead>
<tr>
<th>Altitude, km</th>
<th>$\text{RMS} (\Delta T)/\Delta T_m$ values, %</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>300</td>
</tr>
<tr>
<td>$\sigma_z = 0.0002 \text{ min}$</td>
<td>10</td>
</tr>
<tr>
<td>$\sigma_z = 0.001 \text{ min}$</td>
<td>6.5</td>
</tr>
</tbody>
</table>

It is seen from these results that the relative errors of determining the mean drag on the fit span lie in the range of 5-10%. The value of errors highly depends on the fit span duration. This is just the reason, why they decrease with increasing altitude and worsening the accuracy of measurements.

6. Effect of increasing time interval between measurements

We shall consider the results of estimating the RMS of time errors of forecasting for the following values of parameters:

Time interval between the measurements (in revolutions) $dN=8;$
The increase of the interval between measurements from 2 to 8 revolutions is essential here.

The possible range of altitudes of satellites affected by drag in the atmosphere was accepted to be 300 – 800 km (as in Section 4). The results of determining the optimum fit span with using LST are presented in Table 5. The estimates of the optimum fit span (in days) are shown by a separate line. In all remaining aspects the form of this table is similar to that of Table 2.

These results are characterized by the fact, that in the majority of cases the optimum fit span values have reached a limitation on its lowest value (4 measurements). In this case the optimum fit span value occurred to be essentially greater (up to 3 times), than that for the same altitudes according to Table 2 (at dN=2). As a result, the level of disturbances over the measurement interval has increased for these versions. It should be expected that the level of errors would increase as well. And only for SOs with altitudes of 700 and 800 km, the number of measurements over the measurement interval occurred to be greater than 4, and its duration was rather well correlated with the corresponding data of Table 2.

The estimates of RMS of errors at the initial point (at the last measurement instant), as well as at forecasting for 15 and 60 revolutions are presented in Table 6, whose form is similar to that of Table 3. The same results are presented in Figures 8, 9 and 10.

Table 5. Fit span values (number of measurements/days) for SOs at various altitudes at dN=8

<table>
<thead>
<tr>
<th>Altitude, km</th>
<th>nz values/fit span (days)</th>
<th>ΔT_m, min/rev</th>
<th>ΔT_m, min/day</th>
<th>σ_q⋅ΔT_m, min</th>
<th>nz</th>
<th>Fit span, days</th>
</tr>
</thead>
<tbody>
<tr>
<td>300</td>
<td>4</td>
<td>2.0E-3</td>
<td>0.030</td>
<td>2.0E-4</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>400</td>
<td>4</td>
<td>0.6E-3</td>
<td>0.009</td>
<td>0.6E-4</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>500</td>
<td>4</td>
<td>2.5E-4</td>
<td>0.0037</td>
<td>2.5E-5</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>600</td>
<td>6</td>
<td>1.0E-4</td>
<td>0.0015</td>
<td>1.0E-5</td>
<td>6</td>
<td>3</td>
</tr>
<tr>
<td>700</td>
<td>8</td>
<td>4.0E-5</td>
<td>0.0006</td>
<td>4.0E-6</td>
<td>8</td>
<td>4</td>
</tr>
<tr>
<td>800</td>
<td></td>
<td>1.5E-5</td>
<td>0.00022</td>
<td>1.5E-6</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

RMS of time measurement errors $\sigma_z = 0.0002 \text{ min} = 0.012 \text{ sec}$; Interval of atmospheric noise correlation $\Delta = 50 \text{ revolutions}$; RMS of normalized drag variations $\sigma_q = 0.1$.

The values of RMS of errors at the last measurement instant coincide for five calculation versions from six. And only for the SO altitude of 300 km, the estimates for LST are almost 2 times greater. In this case the level of noises is greatest, and the application of LST did not provide a possibility to well "inscribe" in all measurements. We shall pay attention to the fact, that with using the OFM technique the fit span was rather great (16 days), but this fact did not cause worsening the accuracy of estimates of orbital parameters.
Figure 8. RMS of errors at the last measurement instant, sec
Thus, the materials of the given Section indicate, that in the case of the lack of measurements the application of the OFM technique makes it possible to apply rather large measurement intervals and to obtain, due to more correct calculation of weight matrix, essentially better accuracy as compared to LST.

**7. Effect of increasing the measurement accuracy**

We shall consider the results of estimating the RMS of forecasting time errors for the following values of parameters:

- Time interval between the measurements (in revolutions) \( dN=8 \);
- Mean value of period change per revolution \( \Delta T_m = 0.002 \) min;
- Interval of atmospheric noise correlation \( \Delta = 50 \) revolutions;
- RMS of normalized drag variations \( \sigma_q = 0.1 \).

Possible values of RMS of time measurement errors are accepted in the range of values from \( \sigma_z = 0.000006 \) min to \( \sigma_z = 0.0006 \) min. These values (in various dimensions), the corresponding values of the "signal-to-noise merit" \( S_n = \sigma_q \cdot \Delta T_m / \sigma_z \), as well as the estimates of an optimum number of measurements and corresponding duration of a fit span (for LST) are presented in Table 7. Here, when addressing to LST, the number of realizations was increased up to 5000. For three calculation versions from 5 (with more accurate measurements) the optimum fit span estimates "have entered" the lower limitation (4 measurements).

**Table 7. Characteristics of calculation versions and fit span values**

<table>
<thead>
<tr>
<th>( \sigma_z )</th>
<th>( S_n )</th>
<th>nz</th>
<th>fit span, days</th>
</tr>
</thead>
<tbody>
<tr>
<td>Min</td>
<td>Sec</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.6E-5</td>
<td>0.2E-4</td>
<td>0.6E-4</td>
<td>0.2E-3</td>
</tr>
<tr>
<td>3</td>
<td>10</td>
<td>30</td>
<td>100</td>
</tr>
<tr>
<td>4</td>
<td>4-5</td>
<td>4-5</td>
<td>5</td>
</tr>
</tbody>
</table>
The estimates of RMS of errors at the initial point (at the last measurement instant), as well as at forecasting for 15 and 60 revolutions, are presented in Table 8, whose form is similar to that of Table 3. The same results are presented in Figures 11, 12 and 13.

Table 8. RMS of SO forecasting errors (sec) for various accuracy of measurements

<table>
<thead>
<tr>
<th>$\sigma_z$ min</th>
<th>LST by nz=opt</th>
<th>OFM, nz=30</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Sig(0)</td>
<td>Sig(15)</td>
</tr>
<tr>
<td>0.6E-5</td>
<td>0.00065</td>
<td>0.683</td>
</tr>
<tr>
<td>0.2E-4</td>
<td>0.00128</td>
<td>0.661</td>
</tr>
<tr>
<td>0.6E-4</td>
<td>0.00351</td>
<td>0.653</td>
</tr>
<tr>
<td>0.2E-3</td>
<td>0.0112</td>
<td>0.793</td>
</tr>
<tr>
<td>0.6E-3</td>
<td>0.03191</td>
<td>0.991</td>
</tr>
</tbody>
</table>

These data reveal the following general regularities: 1) the growth of errors of determining and forecasting the orbital parameters with worsening the accuracy of measurements, and 2) the lower level of errors with using the OFM technique as compared to application of LST.

Some interesting regularity is seen from the data of Figure 10. It consists in the fact, that for the "signal-to-noise merit" $S_n = \sigma_q \cdot \Delta T_m / \sigma_z \leq 10$ the RMS of errors at the last measurement instant virtually coincide (for LST and OFM) with the values of RMS of measurement errors ($\sigma_z$). But with a greater value of the $S_n$ ratio the essential (almost two-fold) difference between the LST and OFM technique estimates appears. This implies that, with such a level of noises and application of LST, the model of motion (10) does not allow to "well inscribe" in 4 measurements.

The data of Figures 12 and 13 indicate that, at forecasting, the difference between the solutions of LST and OFM technique reveals itself essentially higher, than for estimations at the last measurement instant. Here, whereas for the OFM technique the presented dependencies are monotonous, for LST estimations the RMS of forecasting errors increase as the $\sigma_z$ values fall below 0.0036 sec. In this region the values of the "signal-to-noise merit" $S_n = \sigma_q \cdot \Delta T_m / \sigma_z \geq 3$. As it was noted above, such a character of behavior of estimates is due to the fact, that in this case the model of motion (10) does not allow to "well inscribe" in 4 measurements.
measurements. For other values of a measurement interval the RMS of errors would be even worse.

Figure 12. RMS of errors at forecasting for 15 revolutions, sec

For the OFM technique the characteristic feature of RMS estimates at forecasting is the non-uniform increase of forecasting accuracy with increasing accuracy of measurements. With 100-fold increasing of measurement accuracy the errors of forecasting for 1 day decreased about 1.7 times. In this case essential improvement of forecasting accuracy takes place at transition from the version of $\sigma_z = 0.036$ sec to the version of $\sigma_z = 0.0036$ sec. At transition from the version of $\sigma_z = 0.0036$ sec to the version of $\sigma_z = 0.00036$ sec the measurement errors decrease an order of magnitude; however, the forecasting errors decrease by about 10% only. With further increasing of measurement accuracy one can hardly hope for essential growth of forecasting accuracy. This result testifies to the fact, that the basic reserve of further growth of forecasting accuracy lies in lowering the level of system’s noises (quantity $\sigma_q$, the "signal-to-noise merit") and transition to applying the OFM technique, rather than in increasing the measurement accuracy.

8. Effect of the interval of atmospheric disturbances correlation
Consider now the influence of the correlation interval \( \Delta \) on the results of estimating the RMS of time errors of forecasting. We remind that the following basic parameters are applied here:

- Time interval between the measurements (in revolutions) \( dN = 2 \);
- Mean value of period change per revolution \( \Delta T_m = 0.002 \) min;
- RMS of normalized drag variations \( \sigma_q = 0.1 \);
- RMS of time measurement errors \( \sigma_z = 0.0002 \) min.

The "signal-to-noise merit" \( S_n = \sigma_q \cdot \Delta T_m / \sigma_z = 1.0 \) corresponds to these values of parameters.

Calculations were carried out for the following values of atmospheric noise correlation interval \( \Delta = 10, 25, 50, 100, 150 \) revolutions. When transferring to LST, the optimum number of measurements was \( nz = 4 \) - 5. This corresponds to the measurement interval of \( \approx 0.5 \) day.

The estimates of RMS of errors at the initial point (at the last measurement instant), as well as at forecasting for 15 and 60 revolutions, are presented in Table 9, whose form is similar to that of Table 3. The same results are presented in Figures 14, 15 and 16.

<table>
<thead>
<tr>
<th>( \Delta ) revolutions</th>
<th>LST by ( nz = \text{opt} )</th>
<th>OFM, ( nz = 30 )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Sig(0)</td>
<td>Sig(15)</td>
</tr>
<tr>
<td>10</td>
<td>0.0112</td>
<td>1.450</td>
</tr>
<tr>
<td>25</td>
<td>0.0114</td>
<td>1.017</td>
</tr>
<tr>
<td>50</td>
<td>0.0117</td>
<td>0.817</td>
</tr>
<tr>
<td>100</td>
<td>0.0112</td>
<td>0.670</td>
</tr>
<tr>
<td>150</td>
<td>0.0118</td>
<td>0.627</td>
</tr>
</tbody>
</table>

These data reveal the general regularities, which are typical for the overwhelming majority of calculation versions: 1) the decrease of errors of determination and forecasting the orbital parameters with increasing correlation interval, and 2) the lower level of errors with using the OFM technique as compared to application of LST. The mentioned regularities are quite expectable. The exceptions from these general regularities are as follows:

- Fluctuations of Sig (0) estimates in the vicinity of the mean value of 0.0115 sec with the amplitude of 0.0005 sec according to the data of Figure 14 for LST, which are explained by a limited number of realizations in statistical modeling.
- With using the OFM technique and for the correlation interval \( \Delta = 10 \) revolutions the estimates of RMS of errors of forecasting for 60 revolutions (12.6 sec) were essentially lower, than RMS of errors (15.6 sec) for the correlation interval \( \Delta = 25 \) revolutions, and essentially lower, than corresponding RMS with using LST (22.6 sec).
Figure 14. RMS of errors at the last measurement instant, sec

Figure 15. RMS of errors at forecasting for 15 revolutions, sec

Figure 16. RMS of errors at forecasting for 60 revolutions, sec
The reasons for the mentioned deviation of RMS of errors from the monotonous dependence with low values of correlation interval ($\Delta = 10$) are not quite clear. The possible explanation lies in the fact that the efficiency of application of the OFM technique depends on the relation between the forecasting interval and correlation interval and increases as this relation grows. Additional investigations are necessary for clarifying the reasons of mentioned deviation of RMS of errors from the monotonous dependence.

9. Influence of a number of measurements upon the forecasting accuracy

Above, in Section 3, in studying the influence of a processing interval on the accuracy of estimations, the main attention was given to the least-square technique. In particular, it was shown that for LST there exists some optimum fit span, for which the minimum values of RMS of errors of determination and forecasting of orbital parameters are achieved. In the process of modeling, rather short processing intervals (up to 22 revolutions) have been considered, that did not allow one to estimate the work of the OFM technique for greater fit spans.

In the given Section the processing intervals up to 200 revolutions were considered. The values of a number of measurements (nz) on the measurement interval were accepted to be equal to 15, 30, 60 and 100. In relation to the basic parameters, the RMS of measurement errors have been varied. Namely, three versions of values of RMS of time measurement errors have been considered: 0.00006, 0.0002 and 0.0006 minutes.

We remind that the following basic parameters are applied here:

- Time interval between the measurements (in revolutions) $dN = 2$;
- Mean value of period change per revolution $\Delta T_m = 0.002$ min;
- RMS of normalized drag variations $\sigma_q = 0.1$.

Atmospheric noise correlation interval $\Delta = 50$ revolutions

The values of the "signal-to-noise merit" $S_n = \sigma_q \cdot \Delta T_m / \sigma_z$ = 3.3, 1.0 and 0.33 correspond to the mentioned three versions of $\sigma_z$ estimations.

The estimates of RMS of the normalized time errors of forecasting for various values of a number of measurements as a function of forecasting interval are presented in Figures 17, 18 and 19. Besides, for each of three versions of $\sigma_z$ values, the RMS of forecasting errors with using LST are presented, which were calculated for the optimum number of measurements, that was accepted to be equal to 4, 5 and 7, respectively.
Figure 17. RMS of time errors of forecasting for $\sigma_z = 0.00006$ min

It is seen from the data of this figure, that application of the OFM technique with the number of measurements $n_z=15$ results, at forecasting for 5-30 revolutions, in slightly greater errors as compared to the results of application of LST. The increase of a number of measurements up to 60-100 results in essential lowering of errors. With forecasting intervals longer than 1 day ($\approx 15$ revolutions) the RMS of errors of the OFM technique is 1.4 times lower, than the corresponding results of the LST application.

Figure 18. RMS of time errors of forecasting for $\sigma_z = 0.0002$ min

As compared to the previous figure, these results correspond to less accurate measurements. The positive effect of the OFM technique application is revealed in all cases. With the number of measurements greater than 60 and with forecasting intervals longer than 1 day ($\approx 15$ revolutions) the RMS of errors of the OFM technique is 1.3 times lower, than corresponding results of LST application.
As compared to the previous figure, these results correspond to further worsening of measurement accuracy. It is seen that the positive effect of the OFM technique application is revealed in all cases. With the number of measurements greater than 60 and with forecasting intervals longer than 1 day (∼15 revolutions), the RMS of errors of the OFM technique is 1.25-1.35 times lower, than corresponding results of LST application. This effect increases with growing fit span. We remind that for LST and under the given conditions the optimum number of measurements equals 7 (which correspond to the measurement interval of 1 day).

In all considered calculation versions the increase of a number of changes results in non-uniform increasing of accuracy. RMS of errors decrease most essentially at transition from 15 to 30 measurements. Further on, especially at transition from 60 to 100 changes, the effect appears to be rather low. The given result testifies to the possibility of lowering the number of measurements with using the OFM technique. In this case the recommended number of measurements are an order of magnitude greater, than with using LST.

In summary we shall show the panel of our program, which was used at calculations.

![Figure 20. Panel of software “LST & OFM errors”](image)

**Conclusions**

1. A comparative analysis of the accuracy of determination of SO’s state vector with accounting for nuisance (disturbing) parameters was performed with various approaches to the problem solution:
   - By means of LST without accounting for nuisance (disturbing) parameters
   - By means of LST with including disturbing parameters into the state vector (parameterization);
   - By means of maximum-likelihood method (MLM) on the basis of accounting for statistical characteristics of disturbing parameters (without parameterization). The a priori correlation matrix of nuisance parameters is used for “weighing” the measurements without extension of a state vector.
It is shown that there exists a level of (small) disturbances, for which it is more profitable to apply the LST without extension of a state vector. However, even in this case the errors are greater, than in case of using the non-parametric approach. In all cases, the latter approach provides the best accuracy (optimum filtering of measurements).

2. The statistical model is developed for estimating time errors of determination and forecasting orbits influenced by atmospheric disturbances of color noise type. The basic influencing factors are considered to be:
   - Standard deviation of measurement errors;
   - Time interval between the measurements;
   - Number of processed measurements (fit span);
   - Standard deviation of color noise;
   - Time interval of color noise correlation.

3. The dependence of errors on basic influencing factors was studied. In so doing, two approaches to state vector estimations were considered: 1) with applying LST and without accounting for disturbing parameters, and 2) on the basis of optimum filtering of measurements with accounting for statistical characteristics of atmospheric disturbances. As a result, the estimates of time errors under various conditions were obtained.

4. The results of the performed investigation can be useful for updating the orbit correction techniques applied in Space Surveillance Systems, which is characterized by extreme diversity of orbits and measurement performance modes. This especially concerns small-size SOs, for which obtaining a great number of measurements is problematic.

References