A SEMIANALYTIC MESHLESS APPROACH TO THE TRANSIENT FOKKER-PERLANCK EQUATION

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Abstract

A semianalytic partition of unity finite element method (PUFEM) is presented to solve the transient Fokker-Planck equation (FPE) for high-dimensional nonlinear dynamical systems. Meshless spatial discretization of the PUFEM is employed to develop linear ordinary differential equations for the time varying coefficients of the local shape functions. A similarity transformation to the modal coordinates is shown to reveal numerous spurious modes in the eigenspace of the discretized FPE operator. The identification and elimination of these modes leads to an analytical solution of the ODEs obtained from the spatial discretization in terms of the remaining admissible modes, and a significant order-reduction in the transient problem. The initial equation-error resulting from the set of admissible modes is shown to be an upper bound for all time, and thus the reduced set is sufficient for the approximation for all time.

Introduction

Uncertainty propagation through dynamical systems has been an important area of research for several decades. The Fokker-Planck equation (FPE) provides the exact description of the uncertainty propagation problem through dynamical systems driven by white-noise. Consider a stochastic dynamical system governed by the following Itô stochastic differential equation (SDE):

\[ dx = f(t, x)dt + g(t, x)dB(t), \quad E[x(t_0)] = \bar{x}_0 \] (1)

where, \( B(t) \) is a \( m \)-dimensional Brownian motion process, and \( f(t, x) : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n \) and \( g(t, x) : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^{n \times m} \) are measurable functions. The state of the system, \( x(t) \), is a \( n \)-dimensional stochastic process whose pdf at any time \( t \) is denoted by \( W(t, x) \) and the initial density is known to be \( W(t_0, x) = W_0(x) \). Then, the time-evolution of the instantaneous pdf of \( x(t) \) is governed by the following well known Fokker-Planck equation:

\[ \frac{\partial}{\partial t} W(t, x) = \left[-\sum_{i=1}^{N} \frac{\partial}{\partial x_i} D^{(1)}_i + \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{\partial^2}{\partial x_i \partial x_j} D^{(2)}_{ij}\right] W(t, x) \] (2)

where,

\[ D^{(1)}(t, x) = f(t, x) + \frac{1}{2} \frac{\partial g(t, x)}{\partial x} Q g(t, x), \quad D^{(2)}(t, x) = \frac{1}{2} g(t, x) Q g^T(t, x) \] (3)

The spatial operator on the RHS of Eq.2 is called the Fokker-Planck operator, denoted by \( L_{FP} \). \( D^{(1)} \) is known as the drift coefficient vector and captures the drift between the propagated mean and mean of the propagated pdf. Such drift increases with the degree of nonlinearity in the underlying dynamics, i.e. \( f(t, x) \). \( D^{(2)} \) is known as the diffusion matrix
and captures the “flattening” of the pdf over time. The expressions in Eq.3 represent the Stratonovich form of the FPE, used extensively in the engineering community over the Itô form, whose derivation and analysis requires the use of stochastic calculus based on Itô’s rules of stochastic differentiation. This paper will deal with the Stratonovich form of the FPE. Analytical solutions to Eq. 2 are known to exist only for linear dynamics. Stationary solutions are additionally known for a small class of nonlinear dynamical systems with a particular Hamiltonian structure (Fuller, 1969). These systems represent a very small class of the systems encountered in engineering and science and thus a numerical approach to the FPE is warranted. However, numerical approaches for Eq.2 have also been plagued by several difficult issues, outlined in detail in Kumar et al. (2006).

Muscolino et al. (1997) presented a global approximation approach using global C-type Gram-Chalier expansions to solve the transient FPE. This method (used in conjunction with a log-transformation to ensure positivity of approximation) provides extremely good results for systems governed by polynomial nonlinearities, largely due to the log-transformation. Local methods (e.g. finite difference (FD) and finite elements (FE)) have been employed by several researchers (Wojtkiewicz et al., Johnson et al., 1997, 2000) and success has been reported for the 4 dimensional FPE recently. In these works, the ODEs obtained from the discretization process are solved numerically with stabilized integration schemes like the Crank-Nicholson algorithm due to the ill-conditioned nature of the system. Paola and Sofi (2002) have employed a modified-norm approach to obtain improved approximation accuracy for the FPE. Similar work has been done by Kumar et al. (2006) for the purpose of domain determination and solution refinement of the FPE.

In the current paper, we employ a meshless finite element approach for the transient FPE. This approach alleviates the burdens typically associated with the traditional FEM and allows for easy extension to problems in higher dimensions. The approach is semianalytic, and involves spatial discretization and Galerkin error projection, followed by the analytical integration of the reduced order system of ODEs obtained from the error-projection process. The spatial discretization is performed under the meshless PUFEM framework, which approximates the time varying pdf in terms of local shape functions. This process results in a system of linear ODEs for the coefficients of the shape functions, which is then integrated analytically using a similarity transformation. A key result is that such a transformation reveals numerous spurious modes in the eigenspace of the discretized FP-operator which have no physical relevance to the problem. The identification and elimination of these extraneous modes leads to a significant order reduction of the transient problem, improved computational efficiency, and most importantly, an analytical solution of the ODEs in terms of the remaining admissible modes. The latter clearly obviates the need for a stabilized numerical time-integration scheme. Furthermore, it is shown that with a given minimal set of admissible eigenfunctions, the equation-error at all times remains bounded by the equation-error at the initial time. In other words, a minimal set of modes that approximates the initial pdf sufficiently well (in terms of equation-error) is enough to approximate the FPE response for all times. In this paper, we clearly illustrate the order-reduction for a 2-D nonlinear oscillator.
Variational Formulation of the FPE using the Meshless PUFEM

The partition of unity (PU) approach to finite elements was developed by Babuška and Melvenk (1997). It falls under the broad category of meshless FEM which significantly reduces the dependence of problem formulation on the domain of solution and inter-element connectivity, leading to an easy extension to higher dimensional problems. The approximation is constructed over several overlapping domains that form a cover for the global solution domain. These subdomains are defined by the compact support of positive weight functions, called PU-pasting functions satisfying a partition of unity over the global domain (traditionally these are the tent-functions). Inside each subdomain, local shape functions are introduced, forming a conforming finite element approximation space. The local approximations are blended together using the pasting functions. The partition of unity property of the pasting functions ensures unbiased blending. In this paper, we employ pasting functions of higher order continuity than the traditional tent-functions. These are the recently developed GLO-MAP weight functions (Junkins et al., 2004). A detailed description of the (meshless) discretization and shape-function selection for the FPE as used in this approach can be found in Kumar et al. (2006).

Here, we present the local variational equations for the transient FPE using the meshless PUFEM approach. The PUFEM approximation of the instantaneous pdf can be written as:

\[ \hat{W}(t, \mathbf{x}) = \sum_{s=1}^{P} \varphi_s(\mathbf{x}) \sum_{j=1}^{Q_s} a_{sj}(t) \psi_{sj}(\mathbf{x}) = \sum_{s,j} a_{sj}(t) \Psi_{sj}(\mathbf{x}) \]  

where, \( a_{sj}(t) \) are the time-varying coefficients of the local shape functions denoted by \( \Psi_{sj}(\mathbf{x}) \), which are simply the product of the PU pasting functions \( \varphi_s(\mathbf{x}) \) and basis functions, \( \psi_{sj}(\mathbf{x}) \). The PU pasting functions have compact support and satisfy a partition of unity over the entire domain, i.e. \( \sum \varphi_s(\mathbf{x}) = 1, \forall \mathbf{x} \in \Omega \). Their purpose is to blend together (with specified smoothness) local approximations in an unbiased manner by virtue of the PU property. Notice that for every \( \varphi_s, s = 1, \ldots, P \), there are \( Q_s \) local shape functions. The compact support of \( \varphi_s \) restricts the domain of influence of the corresponding basis functions, i.e. \( \psi_{sj}(\mathbf{x}), j = 1, \ldots, Q_s \) to the local subdomain. The unknown coefficients, \( a_{sj}(t) \) are then found by substituting the above approximation for \( W(t, \mathbf{x}) \) into the FPE, followed by the projection of the residual error onto a space of test functions, \( V = \{v(\mathbf{x})\} \).

Following the Galerkin approach, the test functions are chosen to be the same as the shape functions, i.e., \( V = \{ \varphi_s(\mathbf{x}) \psi_{sj}(\mathbf{x}) \} \). We thus obtain the following local Galerkin variational form for the n-D FPE over the local subdomain \( \Omega_s \) with boundary \( \Gamma_s \):

\[ \int_{\Omega_s} \frac{\partial}{\partial t}(\hat{W}(t, \mathbf{x})) vd\Omega - \int_{\Omega_s} L_{\text{FPE}}(\hat{W}(t, \mathbf{x})) vd\Omega + \alpha \int_{\Gamma_s \cap \Gamma} (\hat{W}(t, \mathbf{x}) - W_{\Gamma}(t, \mathbf{x})) vd\Gamma = 0 \]  

Notice that the integrals are computed over the local subdomain and the boundary conditions are enforced using a penalty parameter \( \alpha \) over the part of the local boundary that intersects with the global boundary. Although it is ideally desired to have \( W_{\Gamma} = 0 \) on the global domain boundary \( \Gamma \), we implement artificial boundary conditions as \( (W_{\Gamma} \approx 10^{-9}) \) using the penalty parameter \( \alpha \) to avoid numerical difficulties like a singular system of equations. Putting together the projection equations from all the local subdomains, we are led
to the following system of linear ODEs involving the mass matrix \( M \), the stiffness matrix \( K \) and load vector \( f \):

\[
\sum_{s=1}^{P} \sum_{j=1}^{Q_s} \left[ \int_{\Omega_s} \ddot{a}_{sj}(t) \varphi_s(x) \psi_{sj}(x) v d\Omega - \int_{\Omega_s} a_{sj}(t) LFP(\varphi_s(x) \psi_{sj}(x)) v d\Omega + \alpha \int_{\Gamma_s \cap \Gamma} a_{sj}(t) \varphi_s(x) \psi_{sj}(x) v d\Gamma \right] = 0
\]  

(6)

or,

\[
\mathbf{M} \ddot{\mathbf{a}}(t) + \mathbf{K} \mathbf{a}(t) = \mathbf{f}
\]  

(7)

where,

\[
M_{ij} = \int_{\Omega_s} \varphi_k(x) \psi_{kl}(x) \varphi_p(x) \psi_{pq}(x) d\Omega
\]  

(8)

\[
K_{ij} = \int_{\Omega_s} LFP(\varphi_k(x) \psi_{kl}(x)) \varphi_p(x) \psi_{pq}(x) d\Omega + \alpha \int_{\Gamma_s \cap \Gamma} \varphi_k(x) \psi_{kl}(x) \varphi_p(x) \psi_{pq}(x) d\Gamma
\]  

(9)

\[
f_i = \alpha \int_{\Gamma_s \cap \Gamma} \mathcal{W}_\Gamma(t, x) \varphi_p(x) \psi_{pq}(x) d\Gamma
\]  

(10)

where, \( i = \left( \sum_{s=1}^{k-1} Q_s + l \right) \) and \( j = \left( \sum_{s=1}^{p-1} Q_s + q \right) \) and no implicit summation is implied by the repeated parameters. We mention that the penalty parameter \( \alpha \) used above requires minor tuning in order to prevent numerically induced ill-conditioning of the stiffness matrix. Also notice that irrespective of the underlying dynamical system, the norm of the load vector is exceedingly small (\( \approx 10^{-6} \)), because it involves the integration of the pdf over the domain boundary. These facts will be important in the next section when we discuss the emergence of spurious modes and their elimination.

**Transformation to Modal Coordinates**

Notice that in the absence of parametric uncertainty, Eq.7 represents a time-invariant system of linear equations (LTI). It is often instructive to consider the study of the eigenstructure of a LTI system as it provides valuable insight into the problem. However, in the present case the problem involves several complicated numerical issues like ill-conditioning (due to large penalty parameter) and large size. Additionally, the stiffness matrix is neither symmetric nor definite due to the non-normal nature of the FP operator. Nevertheless, extensive research has been conducted for such systems (Saad 1992, van der Vorst 2003) and we can benefit from such studies. Thus considering Eq.7, we look at the following similarity transformation: \( \mathbf{a}' = \mathbf{V}^{-1} \mathbf{a} \), where, \( \mathbf{V} \) is obtained by solving the generalized eigenvalue problem for the system \( (-\mathbf{K}, \mathbf{M}) \), i.e. \( \mathbf{K} \mathbf{v} = -\lambda \mathbf{M} \mathbf{v} \). This leads us to the familiar decoupled form of the original system (Eq.7):

\[
\dot{\mathbf{a}}'(t) = \Lambda \mathbf{a}'(t) + \mathbf{f}',
\]  

(11)
where, $\Lambda$ is a diagonal matrix containing the generalized eigenvalues for $(-K, M)$, and $f' = V^{-1}M^{-1}f$ is the load vector in the modal coordinates. Thus, if we can handle the numerical issues involved in computing $V$ and $\Lambda$, we obtain the transient behavior of the system analytically in the modal space. The time history of the individual modal amplitudes, $a'_i(t)$, can be written as:

$$a'_i(t) = \left( a'_i(t_0) + \frac{f'_i}{\lambda_i} \right) \exp(\lambda_i t) - \frac{f'_i}{\lambda_i}.$$

It was found that given the particular fineness of meshless discretization in use, the solution of the generalized eigenvalue problem of $(-K, M)$ contains several extraneous modes. These modes do not affect the improvement of approximation accuracy and in fact only a small subset of the computed eigenfunctions is sufficient to produce the transient FPE response. We consider these issues in the following section.

**Identification and Elimination of Extraneous Modes**

The solution to the generalized eigenvalue problem of the discretized FPE leads to numerous spurious modes. These extraneous modes in the spectrum of the discretized Fokker-Planck operator can be classified into two groups. The first group is an artifact of the penalty method used for boundary condition enforcement and comprises of eigenfunctions that display severe boundary condition violation. These modes are all either highly stable or unstable (i.e. have large negative or positive real parts), depending on the sign of the penalty parameter. In the former case, the dynamics corresponding to these modes die out almost instantly, and in the latter, cause the corresponding modal amplitudes to diverge. It is in fact easy to prove that a large positive penalty parameter in Eq.5 guarantees the existence of unstable modes (it can be shown that the matrix $-M^{-1}K$ Hurwitz implies that the stiffness matrix $K$ is Hurwitz; a contradiction since the FP operator is non-normal). The eigenvalues belonging to this group of spurious modes are easy to identify by virtue of their large magnitude and unstable nature (for a positive penalty parameter). They appear as a distinct band in the spectrum with the largest absolute values and are easy to isolate.

The second group of spurious modes comprises of unreliable eigenfunctions that have not converged for the particular spatial discretization in use. These eigenfunctions are identified by evaluating the equation error in the functional eigenvalue problem of the FP operator. In other words, while these functions satisfy the discretized eigenvalue problem exactly, they show large error in the original (non-discretized) eigenvalue problem in function space. This is due to the fact that the spatial discretization in use is unable to capture these eigenfunctions sufficiently well. Thus we look at the following equation error:

$$\| \varepsilon_\psi(x) \| = \| L_{FP}(\psi(x)) - \lambda \psi(x) \|_{L_2(\Omega)}$$

All the unconverged eigenfunctions appear as a distinct band showing high residual error and can be filtered out easily.

Thus eliminating the extraneous modes, we are left with a reduced set ($\mathcal{A}$) of “admissible” eigenfunctions that can be used to approximate the solution of the transient FPE. Notice that these selected modes (set $\mathcal{A}$) work for all possible initial distributions. As a result, once these modes are identified and isolated as a pre-processing step, we can solve the FPE for any given initial distribution in real-time. The identification of spurious modes is presented in the results section.
For a prescribed initial distribution, it was additionally found that it is possible to identify an even smaller set \( B \subset A \), that approximates the transient solution and provides further order reduction. The set \( B \), identified by the process of elimination, is such that it approximates the initial state PDF sufficiently well in terms of the equation-error of the FPE. In other words, the eigenfunctions in set \( B \) are chosen such that the equation error in the FPE is less than a prescribed tolerance at the initial time. It can then be shown that the equation error at all subsequent times remains bounded by the initial equation error and thus, the identified minimal admissible subset \( B \) is sufficient to generate the FPE response for the particular initial distribution considered. This result can be stated as the following theorem:

**Theorem 1.** Let \( A = \{ \psi_i : \text{Real}(\lambda_i) < 0, \| \varepsilon_i \| = \| \mathcal{L}_{FP}(\psi_i) - \lambda_i \psi_i \| < \delta \} \) be the set of stable admissible eigenfunctions. Define the equation error in the FPE as \( e(t) = \left\| \frac{\partial}{\partial t} \mathcal{W}(t, x) - \mathcal{L}_{FP}(\mathcal{W}(t, x)) \right\|_{L_2(\Omega)} \).

If a subset \( B = \{ \psi^*_i : \psi^*_i \in A, i = 1, \ldots, N_B \} \) of \( A \) can be identified such that the initial equation error is within a specified tolerance, i.e. \( e(t_0) < \epsilon \), then the equation error at all subsequent times is bounded by the initial equation error, i.e. \( e(t) < \epsilon \).

**Proof.** We assume that \( \mathcal{W}_I = 0 \), such that \( f = 0 \). This assumption is not restrictive because the load vector is ideally zero, and is set to a small value (\( \| f \| \approx 10^{-6} \)) to avoid numerically induced singularities. Now, the equation error in the FPE at time \( t \) is given by:

\[
e(t) = \left\| \frac{\partial}{\partial t} \mathcal{W}(t, x) - \mathcal{L}_{FP}(\mathcal{W}(t, x)) \right\|_{L_2(\Omega)} \tag{14}
\]

Using the proposed subset \( B \) of admissible eigenfunctions, we have the following expression for the instantaneous PDF:

\[
\mathcal{W}(t, x) = \sum_{i=1}^{\text{card}(B)=N_B} a_i^*(t) \psi^*_i(x).
\]

Substituting this expression in Eq.(14) we have the following developments:

\[
e(t) = \left\| \sum_{i=1}^{N_B} \left[ \frac{\partial}{\partial t} (a_i^*(t) \psi^*_i(x)) - \mathcal{L}_{FP}(a_i^*(t) \psi^*_i(x)) \right] \right\|_{L_2(\Omega)} \tag{15} \]

\[
e(t) = \left\| \sum_{i=1}^{N_B} \left\{ \{ \dot{a}_i^*(t) - \lambda_i a_i^*(t) \} \psi^*_i(x) - a_i^*(t) \varepsilon_i^*(x) \right\} \right\|_{L_2(\Omega)} \tag{16} \]

Noting that the eigenfunctions can be normalized to have unit \( L_2(\Omega) \) norm, we get

\[
e(t) \leq \sqrt{\sum_{i=1}^{N_B} |a_i^*(t)|^2} \tag{17}
\]

The time history of the modal amplitudes \( a_i^*(t) \) (in general complex valued) is given by Eq.(12), from which it is easy to show that \( |a(t)| \leq |a(0)| \). We thus conclude from Eq.(17) that \( e(t) \leq e(0) \leq \epsilon \forall t \geq 0 \). \( \square \)
A corollary of the above theorem is that the equation-error is bounded by a mono-
tonically decaying envelope which has greatest width at the initial time. The same is true
for the modal amplitudes $a_i'(t)$. In the special case of systems that admit a stationary
distribution, only one mode (corresponding to $\lambda = 0$, “stationary mode”) has non-zero am-
plitude as $t \to \infty$. Notice that the above theorem holds for $B = A$, and thus is not specific
to particular initial distributions.

**Benefits of Spectral Analysis**

The most important advantage of the transformation to modal coordinates is that it allows
for a robust solution to the FPE in near-real time, given that the eigenvalue analysis is per-
fomed offline. The pre-processing step of identifying the set of admissible eigenfunctions
makes the approach independent of the initial distribution. The use of eigenfunctions en-
sures that the solution accuracy (equation error) improves with time and the approximation
is at least as good as the approximation of the initial distribution in terms of equation-error.
This is a significant step towards obtaining high-fidelity solutions of the FPE. Furthermore,
the process of removing spurious modes leads to a significant order reduction in the prob-
lem, which is useful for high dimensional applications.

**Results**

Consider the following two dimensional nonlinear oscillator (Muscolino et al. 1997):

$$\ddot{x} + \beta \dot{x} + x + \alpha (x^2 + \dot{x}^2) \dot{x} = g(t) G(t) \tag{18}$$

The above system is known to admit a stationary distribution (Muscolino et al. 1997).
The results of modal analysis for the above system are shown in Figs.1 and 2. Recall that
the spurious modes in ‘group 1’ display severe boundary condition violation and those in
‘group 2’ exhibit large equation-error in the eigenvalue problem of the FP-operator in func-
tion space. The modes belonging to group 1 are clearly distinguishable in Fig.1(a) by the
large magnitude of their eigenvalues. Fig.1(b) shows some examples of these eigenfunc-
tions and the exaggerated violation of boundary conditions is clearly visible. Eigenfunc-
tions of group 1 constitute 25% of the total number of modes. Notice that the eigenvalue
with the smallest magnitude is isolated from the rest of the spectrum (i.e. there exists a
spectral gap). The corresponding eigenfunction is the stationary distribution of the dynam-
ical system under consideration. Furthermore, the fact that this mode is isolated is not a
surprise. The FPE is known to be globally asymptotically stable, meaning that if there ex-
stists a stationary solution, it is unique. Hence, there are no other modes in the neighborhood
of the stationary distribution and this fact indeed emerges from the modal analysis. The
actual extent of spectral gap (i.e. separation from the rest of the modes) varies from system
to system. The spectral gap endows exponential stability to the FPE, whose extent depends
on the dynamical system under consideration.

The modes belonging to group 2 are identified in Fig.2, which shows two distinct bands
of equation-error in the functional eigenvalue problem of the FPE. The uniformly large
magnitude of the errors on this plot (see the scale on the $y$-axis) is due to the noise intro-
duced in computing the second derivative of the approximation. The set $A$ of admissible
modes is shown in Fig. 2 surrounded by an ellipse. The smallest eigenvalue (corresponding to the unique stationary distribution) is highlighted and emerges as the eigenfunction with the best convergence. Fig. 3 shows a possible initial distribution, namely a Gaussian pdf with mean at the origin. For this initial distribution, a much smaller set $B$ was found, (shown with circles in Fig. 2) which approximates the initial condition sufficiently well and contains only 20% of the total number of degrees of freedom originally used. This is a significant order reduction. Notice that Fig. 3 provides a verification of the redundant nature of all the eliminated modes $a_i^*(0)$. In this figure, all the initial modal amplitudes $a_i^*(0)$ have been shown in the complex plane. It is clearly visible that the initial amplitudes of all eliminated modes is negligible. In other words, they do not participate in the approximation of the initial distribution, a necessary condition for their redundant nature. Furthermore, following Theorem 1, we know that $|a_i^*(t)| \leq |a_i^*(0)|$, which in turn implies that these modes do not participate in the approximation at any later time and are therefore redundant. Thus, we see that a significant reduction in the order of approximation is achievable for the transient problem. Fig. 4(a) shows the evolution of the modal amplitudes of all the modes. Notice that all the transient modes decay to zero amplitude, while the amplitude of the static mode attains a non zero, (nearly) steady state value. We mention that the eigenvalue of the stationary mode is computed to have a very small real part due to numerical computations and hence has a finite decay rate. However, the decay rate computed for the static mode is several orders of magnitude slower than the other transient modes and we obtain the stationary behavior for all practical purposes. This fact is visible in Fig. 4(a). Furthermore, Fig. 4(b) confirms the exponential decrease in the equation-error in accordance with Theorem 1. Fig. 5 shows the time evolution of the initial distribution obtained from the analytical integration of the coefficients of the remaining modes and all the chosen initial distributions led to the same stationary distribution, as expected.

To complete our analysis, we consider the following 4 dimensional dynamical system:

$$
\begin{align*}
\dot{x}_1 &= x_3 \\
\dot{x}_2 &= x_4 \\
\dot{x}_3 &= -\mu x_1 r^2 + \eta_1 x_3 + g_1 w_1(t) \\
\dot{x}_4 &= -\mu x_2 r^2 + \eta_2 x_4 + g_2 w_2(t), \quad r^2 = x_1^2 + x_2^2
\end{align*}
$$

where, $w = [w_1, w_2]'$ is the two-dimensional Brownian motion. Figure 6 shows the spectrum of the discretized FP operator for the above system. Clearly, the characteristics are much the same as the two-dimensional system and the spurious modes are easy to isolate. The stationary solution stands out once more as an isolated mode with the smallest magnitude. Additionally, all the spurious modes have negligible initial modal amplitudes and do not participate in the approximation of the initial distributions considered. The remaining modes similar time histories as in Fig. 4(a).

**Conclusions**

A semianalytical approach for solving the transient FPE has been presented in this paper. A similarity transformation to the modal coordinates has been used to eliminate numerous spurious modes which do not relate to any physical aspect of the problem. This step is
shown to lead to an analytical solution and a significant reduction in the number of degrees of freedom of the approximation. Furthermore, the use of the meshless PUFEM discretization makes this approach easily extendable to higher dimensional problems. A major advantage of this approach is that after a pre-processing step (for a given dynamical system), it is possible to solve the FPE for the system for any given distribution of the initial conditions. The expression of the approximation in terms of the eigenfunctions also provides a method for identifying a set of natural basis functions for the FPE from the pool of admissible eigenfunctions, which can be used to exploit the capability of using special (non-polynomial) functions as shape functions in the PUFEM approximation space. The approach discussed in this paper opens a slew of application areas that were previously out of range due to the computational overheads in solving the FPE in real time. For example, the current approach makes it possible to build nonlinear filters with exact FPE-based propagators. This is due to the fact that once the modal analysis is carried out for the system, the admissible eigenfunctions are sufficient to propagate any distribution in real time. The proposed FPE solver can also be utilized to attack nonlinear stochastic optimal control problems by developing a policy iteration based on the backward-Kolmogorov equation to solve the Hamilton-Jacobi-Bellman equation. This problem is currently being explored by the authors to solve stochastic optimal control problems in high dimensions. In conclusion, this approach holds immense promise for providing high fidelity solutions of the FPE for numerous applications.

References


Aug 21-24, Keystone, CO, USA.
(a) Unstable Eigenvalues (of Group 1) Have the Largest Magnitudes and are Easy to Identify.

(b) Some Examples of Group 1 Spurious Modes

Figure 1: Identification of Spurious Modes of the Discretized FP operator: Group 1.

Figure 2: Unconverged Eigenfunctions (Group 2) Appear as a Distinct Band showing Large Equation Error in the Functional Eigenvalue Problem.
Figure 3: Initial Amplitudes of all Modes $\in B^C$ are Almost trivial for the Shown Initial Distribution.

Figure 4: Time History of Modal Coefficients and a Verification of Theorem 1 and the Eigenfunction Corresponding to the Smallest Eigenvalue. Notice all Except the Static Mode Decay to Zero.
Figure 5: Solution to the FPE for a 2-D Nonlinear Oscillator Starting with a Gaussian Initial Condition.

Figure 6: Spectrum for a Four Dimensional System Admitting a Stationary Solution to the FPE.