Robust Formation Design for the Magnetospheric Multiscale Mission using a Stochastic Optimization Approach


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NASA Magnetospheric Multiscale (MMS) Mission

- Study of magnetic reconnection, charged particle acceleration, and turbulence in Earth’s magnetosphere
- Reconnection occurs in the electron diffusion region (EDR), where Sun and Earth magnetic fields interact
- EDR is 3D, so a tetrahedron formation of 4 spacecraft is required to build a model
- Exact size is unknown, so several formation sizes will be used

Figure credits: NASA Goddard Space Flight Center
The NASA MMS Mission

- **Mission Description**
  - Four satellites in a nearly regular tetrahedron throughout a Region of Interest (RoI), defined near apogee
  - Mission of 2 phases, reference apogees of 12 $R_e$ and 25 $R_e$
  - Phase I perigee at 1.2 $R_e$ and apogee at 12 $R_e$, and inclination of 28.5° ($e = 0.81818$)
  - Tetrahedron side lengths of 10, 25, 60, and 160 km

- **Mission requirements:**
  - Quality factor $> 0.7$ for 80% of time in RoI – approximate with average quality factor in RoI $\geq 0.78$
  - Side lengths $> 4$ km at all times, 6 km near perigee – use side length $> 6$ km everywhere instead
Formation Evolution Animation

Formation Configuration

Reference Orbit

Orbit x (km)

Orbit y (km)

Quality Factor x (km)

Time (Fraction of Period)
Previous Work and Robust Design

- Previous Work
  - MMS orbit design by Hughes based on Cartesian state; propagation by numerical integration
  - Gim and Alfriend examined the effect of the initial tetrahedron orientation on performance
  - Fast optimal design by Roscoe et al. using differential mean orbital elements and Gim-Alfriend (G-A) STM
  - Modified along-track drift condition to account for RoI constraints

- Spacecraft are spinning, which limits thruster capability
- Maneuver and navigation errors greatly degrade performance
- Our goal: robust formation design to improve performance in the presence of errors
Motivation

- Formation design problems typically involve some uncertainty.
- In particular, large errors can be introduced by:
  - Maneuvering system
  - Navigation system
  - Dynamic model

- Dynamic model errors can be mitigated by using high-fidelity models, which typically incur extra computational costs.
- Maneuver and navigation errors are a result of imperfect spacecraft systems and cannot be easily reduced.
- Once a formation is designed, these errors mean that it cannot be achieved exactly.
- The problem: long-term formation stability is typically very sensitive to initialization errors.
- Solution strategy: sacrifice some performance in the nominal case to gain robustness with respect to errors.
Deterministic Optimization

- Given a performance index, $f(x, a)$, a function of the design variable, $x$, and a random variable, $a$
- Deterministic/nominal optimization approach
  - Let $a = \mu_a$ and minimize $f_{\text{det}}(x) = f(x, \mu_a)$ with respect to $x$
  - Typically, $a$ will be zero-mean (since it represents an error parameter)
- This approach is practical when
  - Errors are small
  - Required formation lifetimes are short
  - Periodic correction maneuvers are allowed (and inexpensive)
- Equivalent to ignoring errors sources and optimizing nominal performance
Robust Optimization

Expected Value

The expected value of a function, $f(a)$, of a random variable, $a$ is

$$E(f) = \int_{\Omega} f(a) P(a) \, da$$

(1)

where $P(a)$ is the PDF of $a$ and $\Omega$ is its domain.

- Given the same performance index, $f(x, a)$
- Stochastic/robust optimization approach
  - Minimize $f_{\text{rob}}(x) = E(f)$ with respect to $x$
  - Requires the PDF of $a$, but takes into account the error parameter distribution and its effect on the performance index
Single-Variable Example Problem

\[ f(x, a) = \frac{ka^2}{x} + \frac{(x - c)^2}{\sigma^2} \]  

- \( x > 0 \) is the design variable, \( a \) is a zero-mean random variable

**Properties**

- Similar to MMS cost
- \( x \) affects the gradient of \( f \) with respect to \( a \)
- When \( x \) is lower, slope of \( \frac{\partial f}{\partial a} \) is steeper
- As \( x \to \infty \), \( a \) no longer affects \( f \)
Deterministic Minimum

Let $a = \mu_a$ and minimize $f_{\text{det}}(x) = f(x, \mu_a)$ with respect to $x$

First-order condition

$$f'_{\text{det}}(x) = 0 = -\frac{k\mu_a^2}{x^2} + \frac{2(x-c)}{\sigma^2}$$

$$\therefore x_{\text{det}}^* = c \quad (3)$$

$f_{\text{det}}(x_{\text{det}}^*)$ is the minimum value of $f$ when $a$ is considered to be a deterministic parameter

In fact, this is also the lowest possible value of $f$ for any $a$, when $x > 0$
Robust Minimum

- Minimize $f_{rob}(x) = E(f)$ with respect to $x$ using PDF, $P(a)$

  $$E(f) = \int_{-\infty}^{\infty} f(x, a) P(a) \, da$$

  $$= \frac{k}{x} \int_{-\infty}^{\infty} a^2 P(a) \, da + \frac{(x - c)^2}{\sigma^2} \int_{-\infty}^{\infty} P(a) \, da$$

  $$= \frac{k\sigma_a^2}{x} + \frac{(x - c)^2}{\sigma^2}$$  \hspace{1cm} (4)

- First-order condition

  $$f'_{rob}(x) = 0 = -\frac{k\sigma_a^2}{x^2} + \frac{2(x - c)}{\sigma^2}$$

  $\therefore \quad 0 = 2x_{rob}^* - 2cx_{rob}^* - k\sigma^2\sigma_a^2$  \hspace{1cm} (5)

- $x_{rob}^*$ depends only on $\sigma_a$
Deterministic and Robust Sensitivity

Sample Parameters

\[ k = 1 \]
\[ c = 5 \]
\[ \sigma = 10 \]
\[ \sigma_a = 10 \]

Results

\[ x_{det}^* = 5 \]
\[ x_{rob}^* = 18.9 \]
Monte Carlo Analysis

- Compute $f$ at $x_{\text{det}}^*$ and $x_{\text{rob}}^*$ for randomly generated values of $a$
- 5000 cases with $a \sim N(0, \sigma_a^2)$

Figure: Deterministic minimization.

\[ \mu_f = 20.0 \]
\[ \sigma_f = 28.3 \]

Figure: Robust minimization.

\[ \mu_f = 7.2 \]
\[ \sigma_f = 7.5 \]
Riemann Sum Approximation

\[ E(f) = \int_{\Omega} f(x, a) P(a) \, da \]  \hspace{1cm} (6)

- It may not be practical to compute the expectation integral for complicated objective functions

- Approximation using Riemann sum: discretize each component, \( a_i \), of \( a \in \mathbb{R}^m \) at \( N_i \) equally spaced points

\[ E(f) \approx \sum_{a_1}^{N_1} \cdots \sum_{a_m}^{N_m} f(x, a) P(a) \Delta V \]  \hspace{1cm} (7)

- \( \Delta V = \Delta a_1 \cdots \Delta a_m \) is the hypervolume of each element
Introduction

Stochastic Optimization

Nominal Design

Robust Design

Conclusions

Robustness Weighting Parameter

- Performance index can be further refined to control tradeoff between nominal performance and robustness
- Assume each component of $a$ is independent and normally distributed with variance, $\sigma^2$:

$$P(a) = \frac{1}{\sqrt{(2\pi)^m \sigma^2}} \exp \left[ \frac{a^T a}{2\sigma^2} \right]$$

$$f^{*}_{rob}(x) = \sum_{j=1}^{N^3} f(x, a_j) \exp \left[ \frac{a_j^T a_j}{2\sigma^2 w} \right]$$

- Introduce weighting parameter, $w$, and let each $N_i = N$ to yield the robust performance index

$$0 < w < 1 \text{ places more emphasis on nominal performance}$$
$$1 < w < \infty \text{ places more emphasis on robustness}$$
$$\text{Constant coefficients are independent of optimization}$$

Roscoe, Vadali, and Alfriend

Robust Formation Design for MMS using Stochastic Optimization

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Single-Variable Example: Robust Approximation

Sample $a$ at $-3\sigma$, 0, and $3\sigma$, for $w = 1$ and $w = 9$
- $w = 1$ yields $x_{rob}^* = 11.9$
- $w = 9$ yields $x_{rob}^* = 30.9$

**Figure**: $w = 1$ minimization.
- $\mu_f = 8.9$
- $\sigma_f = 11.9$

**Figure**: $w = 9$ minimization.
- $\mu_f = 9.9$
- $\sigma_f = 4.6$
Define the state vector

\[ \mathbf{x}_j = \begin{bmatrix} \mathbf{r}_j \\ \dot{\mathbf{r}}_j \end{bmatrix} \]  \hspace{1cm} (10)

Position and velocity defined in curvilinear LVLH frame

- G-A STM

\[ \mathbf{x}_j(t) = \Sigma(t) \mathbf{D}(t) \phi_e(t, t_0) \delta \mathbf{e}_j \]  \hspace{1cm} (11)

- Initial differential nonsingular mean elements

\[ \delta \mathbf{e}_j = \begin{bmatrix} \delta a_j \\ \delta \theta_j \\ \delta i_j \\ \delta q_{1j} \\ \delta q_{2j} \\ \delta \Omega_j \end{bmatrix}^T \]  \hspace{1cm} (12)
Quality Factor

- Volume Factor

\[ Q_v = \frac{V_a}{V_r} \]  \hspace{1cm} (13)

- \( V_a \): volume of actual tetrahedron formed by formation
- \( V_r \): volume of a regular tetrahedron with the side length = \( \bar{L} \)

- \( Q_s \): quadratic Size Factor

- Quality Factor

\[ Q = Q_v Q_s \]  \hspace{1cm} (14)

- Average \( Q \) (with respect to \( t \)) in the RoI is

\[ \bar{Q}_{RoI} \approx \frac{n^3}{N} \frac{f_2 - f_1}{M_2 - M_1} \sum_{k=1}^{N} \frac{Q_k}{(1 + e \cos f_k)^2} \]  \hspace{1cm} (15)

- Approximation of \( Q \) requirement is \( \bar{Q}_{RoI} \geq 0.78 \)
Nominal Formation Optimization

- Single-orbit constrained (SOC) optimization
  - Maximize $\bar{Q}_{RoI}$ over a single orbit subject to
  
  $$
  \delta a_j = \frac{J_2 R_e^2 (3\eta + 4k_j)}{2a_\eta^4} \left[ (3\cos^2 i - 1) \frac{e\delta e_j}{\eta^2} - \sin 2i \delta i_j \right]
  $$

- Multi-orbit unconstrained (MOU) optimization
  - Maximize $\bar{Q}_{RoI}$ over multiple orbits, no constraint

![Figure: Comparison of average quality factor for 10 km formation.](image)
Robust Formation Optimization

- Expectation value for $\bar{Q}_{RoI}$, given PDF, $P(\delta a_{e1}, \delta a_{e2}, \delta a_{e3})$, of the deputies’ $\delta a$ errors

$$E(\bar{Q}_{RoI}) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \bar{Q}_{RoI} P(\delta a_e) \, d\delta a_e$$  \hspace{1cm} (16)

$$\delta a_e = \begin{bmatrix} \delta a_{e1} \\ \delta a_{e2} \\ \delta a_{e3} \end{bmatrix}^T$$  \hspace{1cm} (17)

- Robust cost function

$$J_r = -\sum_{i=1}^{N^3} \bar{Q}_{RoI,i} \exp \left[ -\frac{\delta a_{e,i}^T \delta a_{e,i}}{2\sigma^2 w} \right]$$  \hspace{1cm} (18)

- Robust optimization:
  - Minimize $J_r$ over multiple orbits, subject to modified along-track drift constraint
Performance in the Presence of Errors

- Define $T$ as time until mission requirement violation
- Monte Carlo: random $\delta a$ errors ($3\sigma = 80$ m), 5000 cases
- Desire 14 days minimum between maneuvers (1 orbit $\approx$ 1 day)

![Monte Carlo Analysis of $\delta a$ Errors](image1)

**Figure**: Nominal design. $\bar{T} = 30.6$

$$P (T \leq 14) = 0.082$$

![Monte Carlo Analysis of $\delta a$ Errors](image2)

**Figure**: Robust design, $w = 9$. $\bar{T} = 44.2$

$$P (T \leq 14) = 0.004$$
Performance in the Presence of Errors

- Comparison of sensitivity of $T$ to $\delta a$ errors
- Errors applied to 1 deputy at a time
- Curves represent best case performance: when errors are applied to all deputies, performance will lie below these curves

Figure: Nominal design.

Figure: Robust design, $w = 9$. 
Conclusions

- Robust results have narrower peak to quality factor
- Robust results also have poorer nominal performance in average quality factor

Figure: Quality factor.

Figure: Average quality factor.