Analytical Non-Linear Uncertainty Propagation: Theory And Implementation

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The Relay 2 communication satellite, below, was launched by the United States on Jan. 21, 1984 and stopped working three years later.

The Hubble Space Telescope launched in 1990, orbits at an altitude of 380 miles.

In service from 1963 to 1977, the U.S.'s Micas 6 satellite was designed to detect launched intercontinental ballistic missiles.

Rocket debris, pictured at right, from when the Atlantis space shuttle launched a satellite from space in 1991.

Debris from rocket used to launch Relay 2 left.

Commercial communications satellite launched in June 1999.

Former Soviet Union communication satellite, launched in 1986 and now broken.

International Space Station

Part of Russia's version of the Global Positioning System. this satellite was launched in 1991.

http://www.nytimes.com/2007/02/06/science/20070206_ORBIT_GRAPHIC.html?_r=1
One topic of recent interest in Space Situational Awareness (SSA) and Orbit Determination (OD) is the consistent representation of an observed object’s uncertainty.
Introduction
The proposed technique

- Special solution to the Fokker-Planck eqn. for deterministic systems
  - Non-conservative forces

- State transition tensor (STT)
  - King-Hele’s analytic theory of motion under atm. drag

Analytic expression of the uncertainty for all time
Analytically expressed pdfs can be incorporated in many practical tasks in space situational awareness (SSA)

- **Rapid propagation of the mean and covariance of the uncertainty for arbitrary initial pdfs**
  - Does not involve any sampling
  - Accuracy determined a priori for two-body approximation

- **Model parameter uncertainty readily implemented**
  - Addition of ballistic coefficient and its corresponding uncertainty in the state vector

- Use in non-linear Bayesian estimator for orbit determination and conjunction assessment
For a system that satisfies the Itô stochastic differential equation, the time evolution of a pdf $p$ over $X$ at time $t$ is described as:

$$\frac{\partial p(X,t)}{\partial t} = - \sum_{i=1}^{n} \frac{\partial}{\partial X_i} \{p(X,t)f_i(X,t)\} + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2}{\partial X_i \partial X_j} \left[ p(X,t) \left\{ G(X,t)Q(t)G^T(X,t) \right\}_{ij} \right]$$

As a special solution for deterministic systems, the probability is an integral invariant:

$$\Pr(X \in \mathcal{B}) = \int_{\mathcal{B}} p[X(t)]dX = \int_{\mathcal{B}^0} p[\phi(t; X^0, t^0)] \left| \frac{\partial X}{\partial X^0} \right| dX^0 = \int_{\mathcal{B}^0} p(X^0)dX^0$$
Consequently,

\[ p[\phi(t; X^0, t^0)] = p(X^0) \left| \frac{\partial X}{\partial X^0} \right|^{-1} \]

analytical expr. for the initial pdf
+ analytical expr. for the solution flow
= analytical expr. for the pdf for all time

Propagation is only a matter of changing the value for time
The term accounts for the change in phase volume due to non-conservative forces. For Hamiltonian systems, \( |\frac{\partial \mathbf{X}}{\partial \mathbf{X}_0}|^{-1} = 1 \) from Liouville’s theorem, and therefore:

\[
p[\phi(t; \mathbf{X}^0, t^0)] = p[\mathbf{X}(t)] = p(\mathbf{X}^0) = p[\psi(t, \mathbf{X}; t^0)]
\]

analytical expr. for the initial pdf  
+ analytical expr. for the solution flow  
= analytical expr. for the pdf for all time  

Propagation is only a matter of changing the value for time
Some dynamical systems do not have a closed-form solution, but we can nevertheless express the solution analytically in approximate form via the **State Transition Tensor** (STT) concept.

Take a Taylor series expansion of $\phi(t; X^0, t^0)$ about a reference trajectory $X^*$ ($x = X - X^*$) [Park and Scheeres, 2006]:

$$x_i(t) = \sum_{p=1}^{m} \frac{1}{p!} \Phi_{i,k_1...k_p} x_{k_1}^0 \ldots x_{k_p}^0$$

STTs are a generalization of the state transition matrix to any arbitrary order:

$$x_i(t) = \Phi_{i,k_1} x_{k_1}^0 \Rightarrow x(t) = [\Phi] x^0$$
If \( \phi(t; X^0, t^0) \) known analytically, 
\[
\Phi_{i,k_1...k_p} = \frac{\partial^p X_i}{\partial X^0_{k_1} \ldots \partial X^0_{k_p}}
\]

Otherwise, solve differential equation:
\[
\dot{\Phi}_{i,k_1...k_p} = G(A_{i,k_1}, \ldots, A_{i,k_1...k_p}; \Phi_{i,k_1}, \ldots, \Phi_{i,k_1...k_p})
\]
where \( A \) is the local dynamics tensor:
\[
A_{i,k_1...k_p} = \frac{\partial^p f_i}{\partial X_{k_1} \ldots X_{k_p}}
\]
and ICs: 
\[
\Phi_{i,a} = 1 \quad (i = a)
0 \quad \text{otherwise}
\]
Can also analytically propagate the mean \( \mathbf{m}(t) \) and variance-covariance matrix \( [P] \) of the pdf:

\[
\mathbf{m}_i(t) = \sum_{p=1}^{m} \frac{1}{p!} \Phi_{i,k_1...k_p} \int_{\infty} p(x^0) \left[ \frac{\partial \mathbf{X}}{\partial \mathbf{X}^0} \right]^{-1} x_{k_1}^0 \ldots x_{k_p}^0 \, dx^0
\]

\[
[P]_{ij}(t) = \left[ \sum_{p=1}^{m} \sum_{q=1}^{m} \frac{1}{p!q!} \Phi_{i,k_1...k_p} \Phi_{j,l_1...l_q} \int_{\infty} p(x^0) \left[ \frac{\partial \mathbf{X}}{\partial \mathbf{X}^0} \right]^{-1} x_{k_1}^0 \ldots x_{k_p}^0 x_{l_1}^0 \ldots x_{l_q}^0 \, dx^0 \right] - \mathbf{m}_i(t)\mathbf{m}_j(t)
\]
Can also analytically propagate the mean $m(t)$ and variance-covariance matrix $[P]$ of the pdf:

$$m_i(t) = \sum_{p=1}^{m} \frac{1}{p!} \Phi_{i,k_1...k_p} E \left[ x_{k_1}^{0} \ldots x_{k_p}^{0} \right]$$

$$[P]_{ij}(t) = \left( \sum_{p=1}^{m} \sum_{q=1}^{m} \frac{1}{p!q!} \Phi_{i,k_1...k_p} \Phi_{j,l_1...l_q} E \left[ x_{k_1}^{0} \ldots x_{k_p}^{0} x_{l_1}^{0} \ldots x_{l_q}^{0} \right] \right) - m_i(t)m_j(t)$$

where $E[\cdot]$ is the expected value operator
Example I
Propagation of 3-σ ellipsoid for object in circular LEO influenced by atmospheric drag

• Initial 300 km altitude, 15 deg inclination
• Propagation over 6.286 days for two epochs reflecting a difference in geomagnetic activity

_Calm_ (Feb. 8, 2009)
_Stormy_ (Jul. 11, 2000)
Implementation

Simulation setup (I)

Monte Carlo

- Numerical integration of the dynamics
  \[ \ddot{r} = -\frac{\mu}{r^3} r - \frac{1}{2BC}\rho v v \]

- 2000 sample points distributed normally: \( \sigma_a = 20 \text{ km}, \sigma_M = 0.01 \text{ deg} \)

- Ballistic coefficient allowed to fluctuate sinusoidally
  \[ BC(t) = 10^2 \text{ kg/m}^2 + A_{BC} \sin\left(\frac{2\pi}{\tau_{BC}} t + \phi_{BC}\right), \]

- Density modeled with NRLMSISE-00
Implementation

Simulation setup (I)

Analytic

• 1000 points taken on the 3-σ ellipsoid of the initial Gaussian distribution
  – \( \sigma_a = 20 \text{ km}, \sigma_M = 0.01 \text{ deg}, \sigma_{BC} = 5 \text{ kg/m}^2 \)

• Analytic solution of King-Hele expanded up to 4\textsuperscript{th} order assuming that the atmosphere is…
  – spherically symmetrical
  – density model is exponential with constant parameters in time
  – rotates at the same angular rate as the Earth
Implementation
Applying King-Hele’s results

For an initially circular orbit, only the semi-major axis, and consequently the mean anomaly, change with time

\[ a(t; a^0) = a^0 + H \ln \left[ 1 - \frac{2\pi \delta \rho^0 (a^0)^2 t}{HT^0} \right] = a^0 + H \ln(1 + \epsilon t) \]

\[ M(t; a^0, M^0) = M^0 + \int_0^t n(t) d\tau = M^0 + \int_0^t \sqrt{\frac{\mu}{a(\tau)^3}} d\tau \]

The non-zero terms in the \( n \)-th order STT are

\[ \frac{\partial^n a}{(\partial a^0)^n} \quad \frac{\partial^n M}{(\partial a^0)^n} \quad \frac{\partial M}{\partial M^0} \]
Consider an object with the following parameters:

\[ a^0 = 6678.1 \text{ km}, \quad i^0 = 0.26179 \text{ rad}, \quad e^0 = 0 \]
\[ \omega_E = 7.2722 \times 10^{-5} \text{ rad/s}, \quad \mu = 398600 \text{ km}^3\text{s}^{-2}, \]
\[ H = 40 \text{ km}, \quad \rho^0 = 10^{-11} \text{ kg/m}^3, \quad \text{BC} = 10^2 \text{ kg/m}^2 \]
\[ \Leftrightarrow \varepsilon \sim 10^{-8} \text{ s}^{-1} \]

The integral can be taken analytically by a Taylor series expansion in \( n(t) \) with respect to \( \varepsilon = 0 \)

\[
\frac{n(t)}{\sqrt{\mu}} = \frac{1}{(a^0)^{3/2}} - \frac{3H\varepsilon t}{2(a^0)^{5/2}} + O(\varepsilon^2) \Leftrightarrow M \approx M^0 + \sqrt{\frac{\mu}{(a^0)^3}} t - \frac{3H\varepsilon}{4} \sqrt{\frac{\mu}{(a^0)^5}} t^2
\]
The Jacobian determinant is

\[
\left| \frac{\partial \mathbf{X}}{\partial \mathbf{X}^0} \right| = \frac{\partial a}{\partial a^0} = 1 + \frac{N_1/N_2 - 1}{N_3/N_2 t^{-1} + 1}
\]

For the parameters previously given,

\[
N_1/N_2 \sim 10^{-3}, \ N_3/N_2 \sim 10^7 \ [s] \Leftrightarrow \left| \frac{\partial \mathbf{X}}{\partial \mathbf{X}^0} \right| \approx 1
\]

Thus, the system can be simplified as a Hamiltonian.

Note that these results are easily transformed into non-singular Poincaré elements

\[
\mathcal{L}(t; \mathcal{L}^0) = \sqrt{\mu \cdot a(t; \mathcal{L}^0)}
\]

\[
I(t; \mathcal{L}^0, I^0) = \Omega^0 + \omega^0 + M(t; \mathcal{L}^0, I^0)
\]
Implementation

Results (I)
Calm

• 3-σ curve converges upon the distribution of MC points as the order of the STT is increased

Stormy

• Uncertainties of stochastic drag force are sufficiently expressed through parametric errors
Example II
Compare results of mean and covariance propagation after 1.257 days using:

**Monte Carlo**
- Compute the mean and variance-covariance matrices based on numerical Monte Carlo results

**Analytic**
- Approximate as a Hamiltonian system (i.e. energy is conserved)
- Analytic solution expanded up to 4th order
- **No sampling required**
Relative Error (Truth = MC)

<table>
<thead>
<tr>
<th>Scenario</th>
<th>$a$</th>
<th>$M$</th>
<th>$\sigma_a^2$</th>
<th>$\mu_{aM}$</th>
<th>$\sigma_M^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Calm</td>
<td>0.0127%</td>
<td>-0.0145%</td>
<td>-0.5417%</td>
<td>-0.1815%</td>
<td>-0.3042%</td>
</tr>
<tr>
<td>Stormy</td>
<td>0.0004%</td>
<td>-0.0220%</td>
<td>-0.7657%</td>
<td>-0.5939%</td>
<td>-0.4225%</td>
</tr>
</tbody>
</table>

- Overall, the analytically propagated cumulants are within 1% relative error for all coordinate directions.

- Covariance can be thought of as a direct measure of the “volume” of the uncertainty
  - More strongly influenced by the phase volume contraction than the mean.
Total Runtime (averaged over 10 runs)

<table>
<thead>
<tr>
<th>Monte Carlo</th>
<th>Analytic</th>
</tr>
</thead>
<tbody>
<tr>
<td>23700 s</td>
<td>0.007334 s</td>
</tr>
</tbody>
</table>

- Analytic propagation many orders of magnitude faster than the Monte Carlo
  - Does not scale with propagation time nor sampling resolution of the state space if STT order fixed

- Analytic techniques potentially both accurate and efficient in non-linearly propagating uncertainty
Conclusions

• An analytical method of non-linear uncertainty propagation including non-conservative effects was discussed
  – A special solution to the Fokker-Planck equations for deterministic systems
  – State transition tensor concept

• Applied King-Hele’s theory on orbital motion under atmospheric drag

• Compared with a numerical Monte Carlo simulation with realistic parameter model
Conclusions

Future work

- Further incorporate additional perturbations relevant to Earth-orbiting objects
  - \( J_2 \) added with “semi-empirical” method (Parks, 1983)
  - Better (semi-)analytical theories exist
Questions?

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